

Precise analytic treatment of Kerr and Kerr-(anti) de Sitter black holes as gravitational lenses.

G. V. Kraniotis*

The University of Ioannina, Department of Physics,
Section of Theoretical Physics, GR-451 10 Ioannina, Greece.

September 28, 2010

Abstract

The null geodesic equations that describe motion of photons in Kerr spacetime are solved exactly in the presence of the cosmological constant Λ . The exact solution for the deflection angle for generic light orbits (i.e. non-polar, non-equatorial) is calculated in terms of the generalized hypergeometric functions of Appell and Lauricella.

We then consider the more involved issue in which the black hole acts as a ‘gravitational lens’. The constructed Kerr black hole gravitational lens geometry consists of an observer and a source located far away and placed at arbitrary inclination with respect to black hole’s equatorial plane. The resulting lens equations are solved elegantly in terms of Appell-Lauricella hypergeometric functions and the Weierstraß elliptic function. We then, systematically, apply our closed form solutions for calculating the image and source positions of generic photon orbits that solve the lens equations and reach an observer located at various values of the polar angle for various values of the Kerr parameter and the first integrals of motion. In this framework, the magnification factors for generic orbits are calculated in closed analytic form for the first time. The exercise is repeated with the appropriate modifications for the case of non-zero cosmological constant.

1 Introduction

The issue of the bending of light (and the associated phenomenon of gravitational lensing) from the gravitational field of a celestial body (planet, star, black hole, galaxy) has been a very active and fruitful area of research for fundamental

*Email: gkraniot@cc.uoi.gr.

physics. The Munich astronomer Johann Georg von Soldner in 1801 [1] using Newtonian mechanics and assuming a corpuscular theory for light derived a value for the deflection angle in the Sun's gravitational field $\Delta\phi^S = 0''.875$. Later, Einstein using the equations of general relativity [2], derived a value of $\sim 1''.75$ which is consistent with the findings of Eddington's solar eclipse experiment and subsequent measurements.

Despite the importance of the gravitational bending of light, in unravelling the nature of the gravitational field and its cosmological implications not many exact analytic results for the deflection angle of light orbits from the gravitational field of important astrophysical objects are known in the literature.

Recently, progress has been achieved [6] in obtaining the closed form (strong-field) solution for the deflection angle of an equatorial light ray in the Kerr gravitational field (spinning black hole, rotating mass). Thus, going beyond the corresponding calculation for the static gravitational field of a Schwarzschild black hole [3],[4]. More specifically, the closed form solution for the gravitational bending of light for an equatorial photon orbit in Kerr spacetime was derived and expressed elegantly in terms of Lauricella's hypergeometric function F_D . It was then applied, to calculate the deflection angle for various values of the impact parameter and the spin of the galactic centre black hole Sgr A*. The results exhibited clearly, the strong dependence of the gravitational bending of light, on the spin of the black hole for small values of impact parameter (frame dragging effects) [6]. In addition, in [6], the exact solution for (unstable) spherical bound polar and non-polar photonic orbits was derived. However, the closed form analytic solution for the important class of generic (i.e. non-polar and non-equatorial) unbound light orbits were left out of the discussion in [6].

One of the unsolved related important problems so far was the full analytic treatment of the Kerr and Kerr-de Sitter black holes as gravitational lenses. The closed form solution of this problem is imperative since the Kerr black hole acts as a very strong gravitational lense and we may probe general relativity, through the phenomenon of the bending of light induced by the space time curvature of a spinning black hole, at the strong gravitational field regime. A completely unexplored region of paramount importance for fundamental physics and cosmology.

It is therefore the purpose of the present paper to calculate the exact solution for the deflection angle for a generic photon orbit in the asymptotically flat Kerr spacetime therefore generalizing the results in [6] and solve in closed analytic form the more involved problem of treating the rotating black hole as a gravitational lense. The constructed Kerr black hole gravitational lens geometry consists of an observer and a source located far away and placed at arbitrary inclination with respect to black hole's equatorial plane.

More specifically, we solve for the *first time* in closed analytic form, the resulting lens equations in Kerr geometry, in terms of the Weierstraß elliptic function $\wp(z)$, equation (85), and in terms of generalized hypergeometric functions of Appell-Lauricella equations (89), (82), (60), (99).

In addition, we calculate for the *first time* exactly the resulting magnification factors for generic light orbits in terms of the hypergeometric functions of

Appell and Lauricella. Our closed form solutions for the source and image positions of the lens equations and the corresponding magnification factors represent an important progress step in the extraction of the phenomenological and astrophysical implications of spinning black holes.

The resulting theory is of paramount importance for the galactic centre studies given the strong experimental evidence we have from observation of stellar orbits and flares, that the Sagittarius A* region, at the galactic centre of Milky-Way, harbours a supermassive rotating black hole with mass of 4 million solar masses [7], [8]. Given the fact that the GRAVITY experiment [8] and the proposed 30 metre telescope (TMT) [7] aim at an accuracy of $10 \mu\text{arcs}$ the images calculated in this work and formed near the event horizon of the spinning black hole should be a subject of experimental scrutiny.

Previous efforts on the issue of gravitational lensing from a Kerr black hole were concentrated on various approximations as well as numerical techniques using formal integrals [10],[11].

The material of this work is organized as follows: in section 2, we present the null geodesics in a Kerr spacetime with a cosmological constant. In section 3 we describe the Kerr-lens geometry and relate the first integrals of motion to the observer's image plane coordinates. In section 4 we derive a formal expression for the magnification using the Jacobian that relates observer's image plane coordinates to the source position. This expression involves derivatives of the lens equations in Kerr geometry and in our contribution we shall calculate in closed form these derivatives in terms of the generalized hypergeometric functions of Appell-Lauricella. In section 5, we derive constraints from the condition that a photon escapes to infinity and it is not caught in a (unstable) spherical orbit. These constraints on the Carter's constant and impact factor define a region usually called the shadow of the rotating black hole. For values of the initial conditions inside the region enclosed by the boundary of the shadow and the line with null value for Carter's constant there is no lensing effect since the photons cannot escape and reach an observer. We also discuss constraints arising from the polar motion. In section 7, we derive for the first time the closed form solution for the angular integrals involved in the gravitational Kerr lens, in terms of the generalized hypergeometric functions of Appell-Lauricella. The full exact solution for a light ray which originates from source's polar position and involves m -polar inversions before reaching the polar coordinate of the observer is derived: equations (64),(60). In section 7, we perform the analytic computation of the radial integrals involved in the Kerr-lens in terms of the hypergeometric functions of Appell and Lauricella. In the same section, we derive the closed form solution for the source polar position in terms of the Weierstraß elliptic function $\wp(z, g_2, g_3)$ that implements the constraint that arises from the first lens equation (4). In sections 8, 8.3, we apply our exact solutions for the calculation of the source and image positions for various values of the spin of the black hole and the first integrals of motion, for an equatorial observer and an observer located at a polar angle of $\pi/3$ respectively. We exhibit the image positions on the observer's image plane. In Appendix A, we collect the definition and the integral representation of Lauricella's multivariable hypergeometric function F_D .

In addition in appendix A, we prove in the form of Propositions, some mathematical results concerning the transformation properties of the function F_D which are used in the main text.

2 Null geodesics in a Kerr-(anti) de Sitter black hole.

Taking into account the contribution from the cosmological constant Λ , the generalization of the Kerr solution is described by the Kerr-de Sitter metric element which in Boyer-Lindquist (BL) coordinates is given by [12]-[13]:

$$ds^2 = \frac{\Delta_r}{\Xi^2 \rho^2} (cdt - a \sin^2 \theta d\phi)^2 - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} (acdt - (r^2 + a^2) d\phi)^2 \quad (1)$$

$$\Delta_\theta := 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta, \quad \Xi := 1 + \frac{a^2 \Lambda}{3} \quad (2)$$

$$\Delta_r := \left(1 - \frac{\Lambda}{3} r^2\right) (r^2 + a^2) - 2 \frac{GM}{c^2} r \quad (3)$$

We denote by a the rotation (Kerr) parameter and M denotes the mass of the spinning black hole.

The relevant null geodesic differential equations for the calculation of the gravitational lensing effects (lens-equation) and for the calculation of the deflection angle are:

$$\int^r \frac{dr}{\pm \sqrt{R}} = \int^\theta \frac{d\theta}{\pm \sqrt{\Theta}} \quad (4)$$

$$\Delta\phi = \int d\phi = \int^\theta -\frac{\Xi^2}{\pm \Delta_\theta \sin^2 \theta} \frac{(a \sin^2 \theta - \Phi) d\theta}{\sqrt{\Theta}} + \int^r \frac{a \Xi^2}{\pm \Delta_r} [(r^2 + a^2) - a\Phi] \frac{dr}{\sqrt{R}} \quad (5)$$

where

$$R := \left\{ \Xi^2 [(r^2 + a^2) - a\Phi]^2 - \Delta_r [\Xi^2 (\Phi - a)^2 + \mathcal{Q}] \right\} \quad (6)$$

and

$$\Theta := \left\{ [\mathcal{Q} + (\Phi - a)^2 \Xi^2] \Delta_\theta - \frac{\Xi^2 (a \sin^2 \theta - \Phi)^2}{\sin^2 \theta} \right\} \quad (7)$$

We also derive the equation related to time-delay:

$$ct = \int^r \frac{\Xi^2 (r^2 + a^2) [(r^2 + a^2) - \Phi a]}{\pm \Delta_r \sqrt{R}} dr - \int^\theta \frac{a \Xi^2 (a \sin^2 \theta - \Phi)}{\pm \Delta_\theta \sqrt{\Theta}} d\theta \quad (8)$$

The parameters Φ, \mathcal{Q} are associated to the first integrals of motion. The former is the impact parameter and the latter is related to the hidden first integral (due to the separation of variables in the corresponding Hamilton-Jacobi partial differential equation (PDE)).

3 The Kerr black hole as a gravitational lens.

3.1 Observer's image plane

Assume without loss of generality that the observer's position is at $(r_O, \theta_O, 0)$. Likewise, for the source we have (r_S, θ_S, ϕ_S) . We also assume in this section that $\Lambda = 0$. In the observer's reference frame, an incoming light ray is described by a parametric curve $x(r), y(r), z(r)$, where $r^2 = x^2 + y^2 + z^2$. For large r this is the usual radial BL coordinate. At the location of the observer, the tangent vector to the parametric curve is given by: $(dx/dr)|_{r_O} \hat{\mathbf{x}} + (dy/dr)|_{r_O} \hat{\mathbf{y}} + (dz/dr)|_{r_O} \hat{\mathbf{z}}$. This vector describes a straight line which intersects the (α, β) plane or *observer's image plane* as it is usually called [9]-[11] at (α_i, β_i) see fig.1.

The point (α_i, β_i) is the point $(-\beta_i \cos \theta_O, \alpha_i, \beta_i \cos \theta_O)$ in the (x, y, z) system. Our purpose now is to relate the α_i, β_i variables to the first integrals of motion Φ, \mathcal{Q} . For this we need to use the equation of straight line in space. A straight line can be defined from a point $P_1(x_1, y_1, z_1)$ on it and a vector $\vec{\epsilon}(\epsilon_1, \epsilon_2, \epsilon_3)$ parallel to it. The analytic equations of straight line are then:

$$\frac{x - x_1}{\epsilon_1} = \frac{y - y_1}{\epsilon_2} = \frac{z - z_1}{\epsilon_3} \quad (9)$$

Applying (9) we derive the equations:

$$\frac{-\beta_i \cos \theta_O - r_O \sin \theta_O}{r_O \cos \theta_O \frac{d\theta}{dr}|_{r=r_O} + \sin \theta_O} = \frac{\alpha_i}{r_O \sin \theta_O \frac{d\phi}{dr}|_{r=r_O}} = \frac{\beta_i \cos \theta_O - r_O \cos \theta_O}{\cos \theta_O - r_O \sin \theta_O \frac{d\theta}{dr}|_{r=r_O}} \quad (10)$$

Solving for α_i, β_i we obtain the equations:

$$\alpha_i = -r_O^2 \sin \theta_O \frac{d\phi}{dr}|_{r=r_O} \quad (11)$$

$$\beta_i = r_O^2 \frac{d\theta}{dr}|_{r=r_O} \quad (12)$$

Now we have from the null geodesics that:

$$\frac{d\theta}{dr}|_{r=r_O} = \frac{\Theta(\theta_O)^{1/2}}{R(r_O)^{1/2}} \quad (13)$$

and

$$\frac{d\phi}{dr}|_{r=r_O} = \frac{\Phi}{\sqrt[2]{R(r_O)} \sin^2(\theta_O)} + \frac{2aGM \frac{r_O}{c^2} - a^2 \Phi}{r_O^2 \left[1 + \frac{a^2}{r_O^2} - \frac{2GM}{r_O c^2} \right]} \frac{1}{\sqrt[2]{R(r_O)}} \quad (14)$$

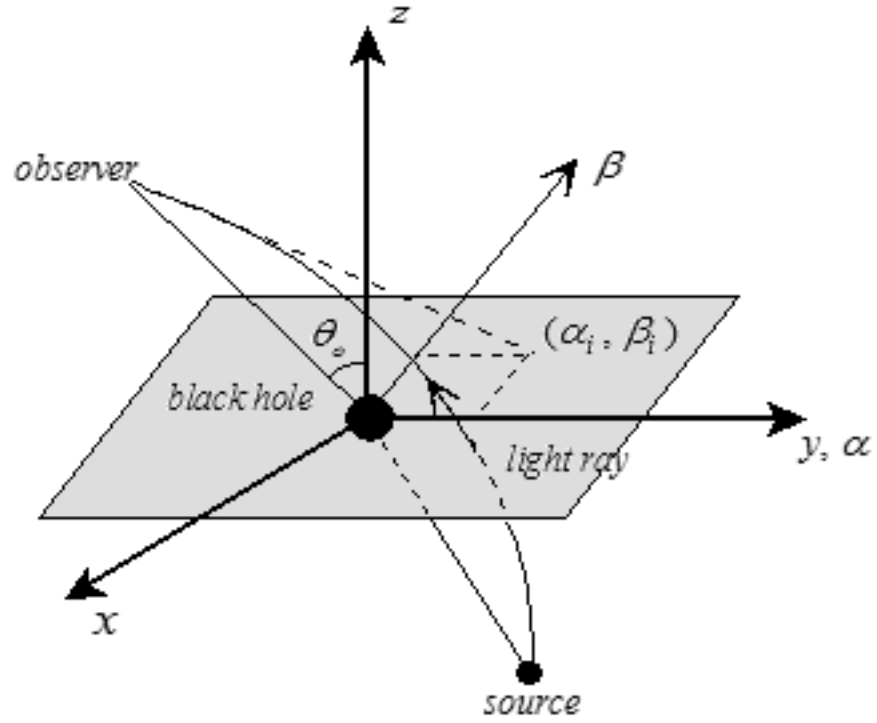


Figure 1: The Kerr black hole gravitational lens geometry. The reference frame is chosen so that, as seen from infinity, the black hole is rotating around the z -axis.

Using eqns(13),(14) and assuming large observer's distance r_O (i.e. $r_O \rightarrow \infty$) we derive simplified expressions relating the coordinates (α_i, β_i) on the observer's image plane to the integrals of motion

$$\Phi \simeq -\alpha_i \sin \theta_O \quad (15)$$

$$\mathcal{Q} \simeq \beta_i^2 + (\alpha_i^2 - a^2) \cos^2(\theta_O) \quad (16)$$

We can also express the position of the source on the observer's sky in terms of its coordinates (r_S, θ_S, ϕ_S) and the observer coordinates. Indeed, the equation for a straight line can be determined by two points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (17)$$

Thus applying the above formula for the straight line connecting the observer and the source yields the equations:

$$\begin{aligned} \alpha_S &= \frac{r_O r_S \sin \theta_S \sin \phi_S}{r_O - r_S (\cos \theta_S \cos \theta_O + \sin \theta_O \sin \theta_S \cos \phi_S)} \\ \beta_S &= \frac{-r_O r_S (\sin \theta_O \cos \theta_S - \sin \theta_S \cos \phi_S \cos \theta_O)}{r_O - r_S (\cos \theta_S \cos \theta_O + \sin \theta_O \sin \theta_S \cos \phi_S)} \end{aligned} \quad (18)$$

4 Magnification factors and positions of images.

In the following sections, we shall perform a detailed novel calculation of the lens effect for the deflection of light produced by the gravitational field of a rotating (Kerr) black hole and a cosmological Kerr black hole (i.e. for non-zero cosmological constant Λ).

The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta\omega$ it subtends on the sky. Since the former quantity is unchanged during light deflection, the ratio of the flux of a sufficiently small image to that of its corresponding source in the absence of the lens, is given by

$$\mu = \frac{\Delta\omega}{(\Delta\omega)_0} = \frac{1}{|J|} \quad (19)$$

where 0-subscripts denote undeflected quantities [5] and J is the Jacobian of the transformation $(x_S, y_S) \rightarrow (x_i, y_i)$ ¹. Writting $x_S = x_S(x_i, y_i), y_S = y_S(x_i, y_i)$ we can find expressions for the partial derivatives appearing in the Jacobian by differentiating equations (4) and (5). Indeed, the Jacobian is given by the expression:

$$J = xw - zy \quad (20)$$

¹Recall in the small angles approximation: $\alpha_i \approx r_O x_i, \beta_i \approx r_O y_i$. Also we define: $x_S := \frac{\alpha_S}{r_O}, y_S := \frac{\beta_S}{r_O}$.

where we defined: $x := \frac{\partial x_S}{\partial x_i}, y := \frac{\partial x_S}{\partial y_i}, z := \frac{\partial y_S}{\partial x_i}, w := \frac{\partial y_S}{\partial y_i}$. Writting equations (4) and (5) as follows:

$$\begin{aligned} R_1(x_i, y_i) - A_1(x_i, y_i, x_S, y_S, m) &= 0 \\ \Delta\phi(x_S, y_S, n) - R_2(x_i, y_i) - A_2(x_i, y_i, x_S, y_S, m) &= 0 \end{aligned} \quad (21)$$

we set up the following system of equations:

$$\beta_1 = -\alpha_1 x - \alpha_2 z \quad (22)$$

$$\beta_2 = -\alpha_1 y - \alpha_2 w \quad (23)$$

$$-\beta_3 = \alpha_3 x + \alpha_4 z \quad (24)$$

$$-\beta_4 = \alpha_3 y + \alpha_4 w \quad (25)$$

where $\alpha_1 = \frac{\partial A_1}{\partial x_S}, \alpha_2 = \frac{\partial A_1}{\partial y_S}, \alpha_3 = -\frac{\partial \phi_S}{\partial x_S} - \frac{\partial A_2}{\partial x_S}, \alpha_4 = -\frac{\partial \phi_S}{\partial y_S} - \frac{\partial A_2}{\partial y_S},$

$$\beta_1 = \frac{\partial R_1}{\partial x_i} - \frac{\partial A_1}{\partial x_i}, \beta_2 = \frac{\partial R_1}{\partial y_i} - \frac{\partial A_1}{\partial y_i}, \beta_3 = \frac{\partial R_2}{\partial x_i} + \frac{\partial A_2}{\partial x_i}, \beta_4 = \frac{\partial R_2}{\partial y_i} + \frac{\partial A_2}{\partial y_i}.$$

Solving for x, y, z, w we obtain:

$$\mu = \frac{1}{|J|} = \left| \frac{\alpha_1 \alpha_4 - \alpha_2 \alpha_3}{\beta_1 \beta_4 - \beta_2 \beta_3} \right| \quad (26)$$

The parameters $n = 0, 1, 2, \dots$ and $m = 0, 1, 2, \dots$ are the number of windings around the z axis and the number of turning points in the polar coordinate θ respectively. We shall discuss the latter in detail in the section that follows.

5 The boundary of the shadow of the rotating black hole and constraints on the parameter space.

The condition for a photon to escape to infinity, which is also the condition for the spherical photon orbits in Kerr spacetime [6], is given by the vanishing of the quartic polynomial $R(r)$ and its first derivative (also in this case $\frac{d^2 R}{dr^2}|_{r=r_f} > 0$). Implementing these two conditions, expressions for the parameter Φ and Carter's constant \mathcal{Q} are obtained [6], [14]:

$$\Phi = \frac{a^2 \frac{GM}{c^2} + a^2 r - 3 \frac{GM}{c^2} r^2 + r^3}{a \left(\frac{GM}{c^2} - r \right)}, \quad \mathcal{Q} = -\frac{r^3 \left(-4a^2 \frac{GM}{c^2} + r \left(\frac{-3GM}{c^2} + r \right)^2 \right)}{a^2 \left(\frac{GM}{c^2} - r \right)^2} \quad (27)$$

The perturbed, from the radius $r = r_{inst}$, of unstable spherical null orbits in Kerr spacetime, and thus escaped photon, will be detected on the observer's

image plane, at the coordinates :

$$\begin{aligned}
x_i &= \frac{a^2(r + \frac{GM}{c^2}) + r^2(r - \frac{3GM}{c^2})}{r_O \sin \theta_O a (r - \frac{GM}{c^2})}, \\
y_i &= \frac{\pm \sqrt{-r^3[r(r - \frac{3GM}{c^2})^2 - 4a^2 \frac{GM}{c^2}] - 2a^2 r(2a^2 \frac{GM}{c^2} + r^3 - 3r \frac{G^2 M^2}{c^4})z_O - a^4(r - \frac{GM}{c^2})^2 z_O^2}}{r_O \sin \theta_O a (r - \frac{GM}{c^2})}
\end{aligned} \tag{28}$$

Equations (28) were derived by plugging into equations (15),(16) the values of the parameters \mathcal{Q}, Φ that correspond to the conditions for the photon to escape to infinity, equations (27). A photon will be detected when the argument of the square root in eqn.(28) is positive. In Eqn(28), $z_O := \cos^2 \theta_O$.

With the aid of equations (15) and (16) we derive:

$$\alpha_i^2 + \beta_i^2 = \Phi^2 + \mathcal{Q} + a^2 z_O \tag{29}$$

Apart from the constraints expressed by equations (27) we also derive constraints for the motion of light from the allowed polar region: $\theta_{\min} \leq \theta_S, \theta_O \leq \theta_{\max}$. Indeed using the variable $z_j := \cos^2 \theta_j$, we have $z_m \geq z_O$ where z_m is the positive root of:

$$-a^2 z_m^2 + (a^2 - \mathcal{Q} - \Phi^2)z_m + \mathcal{Q} = 0$$

Let us see how this can be understood. Defining: $z_m := z_O - x$ we derive the quadratic equation for x

$$-a^2 x^2 - x(a^2 - \mathcal{Q} - \Phi^2 - 2a^2 z_O) - a^2 z_O^2 + z_O(a^2 - \mathcal{Q} - \Phi^2) + \mathcal{Q} = 0 \tag{30}$$

with roots:

$$\begin{aligned}
x_{1,2} &= \frac{-a^2 + \mathcal{Q} + 2a^2 z_O + \Phi^2 \mp \sqrt{4a^2 \mathcal{Q} + (-a^2 + \mathcal{Q} + \Phi^2)^2}}{2a^2} \\
&= \frac{(\alpha_i^2 + \beta_i^2) - a^2 w_O \mp \sqrt{((\alpha_i^2 + \beta_i^2) - a^2 w_O)^2 + 4a^2 \beta_i^2 w_O}}{2a^2}
\end{aligned} \tag{31}$$

where $w_O := \sin^2 \theta_O$. The "radius" $\Phi^2 + \mathcal{Q}$ must be greater or equal than the boundary of the photon region defined by Eqs.(27) and the line $\mathcal{Q} = 0$. The minimum of this value is reached when $\mathcal{Q} = 0$ and $a \rightarrow 1$. The actual minimum value is $(\Phi^2(r) + \mathcal{Q}(r))_{\min} = 4$. Thus, by Eq.(29) we have that $\alpha_i^2 + \beta_i^2 \geq 4$, and since $0 \leq a^2 w_O \leq 1$, it follows the inequality $a^2 w_O - (\alpha_i^2 + \beta_i^2) < 0$ and consequently, $x \leq 0$. Thus we conclude that $z_m \geq z_O$. Similar arguments ensure that when $z_S > z_O$ it follows $z_m \geq z_S$ [11].

6 Closed form solution for the angular integrals.

Let us perform now the exact computation of the angular integrals which occur in the generic photon orbits in Kerr spacetime thereby generalizing the results of [6]. In the case under investigation, we have to take into account the **turning points** in the polar coordinate. A generic angular polar integral can be written:

$$\pm \int_{\theta_1}^{\theta_2} = \int_{\min(z_1, z_2)}^{\max(z_1, z_2)} + [1 - \text{sign}(\theta_1 \circ \theta_2)] \int_0^{\min(z_1, z_2)} \quad (32)$$

where:

$$\theta_1 \circ \theta_2 := \cos \theta_1 \cos \theta_2 \quad (33)$$

Indeed, using the variable $\boxed{z := \cos^2 \theta}$ we derive:

$$-\frac{1}{2} \frac{dz}{\sqrt{z}} \frac{1}{\sqrt{1-z}} = \text{sign}\left(\frac{\pi}{2} - \theta\right) d\theta \quad (34)$$

This is the result of the fact that in the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\cos \theta \geq 0$ and $\sin \theta \geq 0$, while in the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin \theta \geq 0$, $\cos \theta \leq 0$. The angular integration in the polar variable includes the terms:

$$\int^\theta = \pm \int_{\theta_S}^{\theta_{\min/\max}} \pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} \pm \int_{\theta_{\max/\min}}^{\theta_{\min/\max}} \pm \dots \pm \int_{\theta_{\max/\min}}^{\theta_O} \quad (35)$$

The roots z_m, z_3 (of $\boxed{\Theta(\theta) = 0}$) are expressed in terms of the integrals of motion and the cosmological constant by the expressions:

$$z_{m,3} = \frac{\mathcal{Q} + \Phi^2 \Xi^2 - H^2 \pm \sqrt{(\mathcal{Q} + \Phi^2 \Xi^2 - H^2)^2 + 4H^2 \mathcal{Q}}}{-2H^2} \quad (36)$$

and

$$H^2 := \frac{a^2 \Lambda}{3} [\mathcal{Q} + (\Phi - a)^2 \Xi^2] + a^2 \Xi^2 \quad (37)$$

For $\Lambda = 0$, the turning points take the form:

$$\boxed{z_m = \frac{a^2 - \mathcal{Q} - \Phi^2 + \sqrt{4a^2 \mathcal{Q} + (-a^2 + \mathcal{Q} + \Phi^2)^2}}{2a^2}}, \quad (38)$$

where the subscript “m” stands for “min/max”. The corresponding angles are:

$$\boxed{\theta_{\min/\max} = \text{Arccos}(\pm \sqrt{z_m})} \quad (39)$$

Now for θ_j and $\theta_{\min/\max}$ in the same hemisphere:

$$\int_{\theta_j}^{\theta_{\min/\max}} \frac{d\theta}{\pm \sqrt{\Theta(\theta)}} = \frac{1}{2|a|} \int_{z_j}^{z_m} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}} \equiv I_3 \quad (40)$$

Let us now calculate the elliptic integral in eqn.(40) in *closed analytic form*. Applying the transformation:

$$z = z_m + \xi^2(z_j - z_m) \quad (41)$$

our integral is calculated in closed form in terms of Appell's generalized hypergeometric function F_1 of two variables:

$$I_3 = \frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_j)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_j}{z_m}, \frac{z_m - z_j}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \quad (42)$$

On the other hand using the transformation:

$$z = \frac{uz_j z_m - z_j z_m}{uz_j - z_m} \quad (43)$$

we calculate in closed form:

$$\begin{aligned} & \frac{1}{2|a|} \int_0^{z_j} \frac{dz}{\sqrt[2]{z(z_m - z)(z - z_3)}} \\ &= \frac{1}{|a|} \frac{\sqrt[2]{z_j}}{z_m} \sqrt[2]{\frac{z_j - z_m}{z_3 - z_j}} F_1 \left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j}{z_m}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)} \right) \\ &= \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}}}{\sqrt[2]{z_m - z_3}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)} \right) \end{aligned} \quad (44)$$

In going from the second line to the third of (44) we made use of the following identity of Appell's first generalised hypergeometric function of two variables:

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = (1-x)^{-\beta} (1-y)^{\gamma-\alpha-\beta'} F_1(\gamma-\alpha, \beta, \gamma-\beta-\beta', \gamma, \frac{x-y}{x-1}, y) \quad (45)$$

Likewise we derive the closed form solution for the following integral:

$$\begin{aligned} & \frac{1}{2|a|} \int_0^{z_j} \frac{dz}{(1-z) \sqrt[2]{z(z_m - z)(z - z_3)}} \\ &= \frac{z_j}{z_m} \frac{1}{|a|} \frac{z_j - z_m}{1 - z_j} \frac{1}{\sqrt[2]{z_j(z_j - z_m)(z_3 - z_j)}} \times \\ & \quad F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j(1 - z_m)}{z_m(1 - z_j)}, \frac{z_j}{z_m}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)} \right) \\ &= \frac{1}{|a|} \frac{z_j}{z_m} \sqrt[2]{\frac{z_m}{-z_3 z_j}} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z_j, \frac{z_j}{z_m}, \frac{z_j}{z_3} \right) \end{aligned} \quad (46)$$

Producing the last line of equation (46) we used the following formula for the [Lauricella function](#) F_D :

Proposition 1

$$F_D(\alpha, \beta, \beta', \beta'', \gamma, x, y, z) = (1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''} \times F_D\left(\gamma-\alpha, \beta, \gamma-\beta-\beta'-\beta'', \beta'', \gamma, \frac{x-y}{x-1}, y, \frac{z-y}{z-1}\right)$$

Proof. Applying the transformation:

$$u = \frac{1-\nu}{1-\nu y} \quad (47)$$

onto the integral:

$$IR_{F_D} = \int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta'} (1-uz)^{-\beta''} du \quad (48)$$

we derive:

$$\begin{aligned} (1-u)^{\gamma-\alpha-1} &= \left(\frac{\nu(1-y)}{1-\nu y} \right)^{\gamma-\alpha-1}, \quad (1-ux)^{-\beta} = \left(\frac{(1-x)[1-\frac{\nu(x-y)}{(x-1)}]}{1-\nu y} \right)^{-\beta} \\ (1-uy)^{-\beta'} &= \frac{(1-y)^{-\beta'}}{(1-\nu y)^{-\beta'}}, \quad (1-uz)^{-\beta''} = \left(\frac{(1-z)[1-\frac{\nu(z-y)}{(z-1)}]}{1-\nu y} \right)^{-\beta''} \end{aligned} \quad (49)$$

and thus we obtain the result:

$$\begin{aligned} IR_{F_D} &= (1-y)^{\gamma-\alpha}(1-x)^{-\beta}(1-y)^{-\beta'}(1-z)^{-\beta''} \times \\ &\quad \int_0^1 d\nu \nu^{\gamma-\alpha-1} (1-\nu)^{\alpha-1} (1-\nu y)^{-(\gamma-\beta-\beta'-\beta'')} (1-\nu \frac{x-y}{x-1})^{-\beta} (1-\nu \frac{z-y}{z-1})^{-\beta''} \end{aligned} \quad (50)$$

or

$$F_D(\alpha, \beta, \beta', \beta'', \gamma, x, y, z) = (1-y)^{\gamma-\alpha-\beta'}(1-x)^{-\beta}(1-z)^{-\beta''} \times F_D\left(\gamma-\alpha, \beta, \gamma-\beta-\beta'-\beta'', \beta'', \gamma, \frac{x-y}{x-1}, y, \frac{z-y}{z-1}\right)$$

■

Likewise applying (41):

$$\begin{aligned} I_4 &: = \frac{\Phi}{2|a|} \int_{z_j}^{z_m} \frac{dz}{(1-z) \sqrt{z(z_m-z)(z-z_3)}} \\ &= \frac{\Phi}{2|a|} \sqrt{\frac{(z_m-z_j)}{z_m}} \frac{1}{\sqrt{(z_m-z_3)}} \frac{2}{(1-z_m)} F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j-z_m}{1-z_m}, \frac{z_m-z_j}{z_m}, \frac{z_m-z_j}{z_m-z_3}\right) \end{aligned} \quad (51)$$

Let us now compute exactly the term: $\pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}}$, in (35):

$$\pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2 \int_0^{z_m} \quad (52)$$

since $\cos^2 \theta_{\min/\max} = z_m$ and $\theta_{\min} \circ \theta_{\max} = -z_m$.

Equation (51) for $z_j = 0$, becomes :

$$\begin{aligned} & \frac{\Phi}{2|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{2}{(1 - z_m)} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{-z_m}{1 - z_m}, 1, \frac{z_m}{z_m - z_3} \right) \\ = & \frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{1}{(1 - z_m)} \frac{\pi}{2} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{-z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \\ = & \frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{\pi}{2} F_1 \left(\frac{1}{2}, 1, -\frac{1}{2}, 1, \frac{z_m(1 - z_3)}{z_m - z_3}, \frac{z_m}{z_m - z_3} \right) \\ = & \frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{\pi}{2} \frac{1}{1 - z_3} \left(F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) - z_3 F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{z_m(1 - z_3)}{z_m - z_3}, \frac{z_m}{z_m - z_3} \right) \right) \end{aligned} \quad (53)$$

On the other hand the angular integrals of the form $\pm \int_{\theta_S}^{\theta_{\min/\max}}$ in equation (5) are solved in closed analytic form as follows:

$$\begin{aligned} \pm \int_{\theta_S}^{\theta_{\min/\max}} &= \frac{\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1 - z_m)} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1 - z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \\ &+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{z_S - z_m}{1 - z_S} \frac{1}{\sqrt[2]{z_S(z_S - z_m)(z_3 - z_S)}} \times \\ &F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1 - z_m)}{z_m(1 - z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) \end{aligned} \quad (54)$$

An equivalent expression for the above integral is:

$$\begin{aligned}
\pm \int_{\theta_S}^{\theta_{\min/\max}} &= \frac{\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1 - z_m)} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1 - z_m}, \frac{z_S - z_m}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \\
&+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{\Phi}{|a|} \sqrt[2]{\frac{z_S}{z_m}} \sqrt[2]{\frac{z_m - z_3}{z_S - z_3}} \frac{1}{\sqrt[2]{z_m - z_3}} \times \\
&F_D \left(\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{z_S(1 - z_3)}{z_S - z_3}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S}{z_S - z_3} \right) \\
&= \frac{\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1 - z_m)} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1 - z_m}, \frac{z_S - z_m}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \\
&+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{\Phi}{|a|} \sqrt[2]{\frac{z_S}{z_m}} \sqrt[2]{\frac{z_m - z_3}{z_S - z_3}} \frac{1}{\sqrt[2]{z_m - z_3}} \times \\
&\left[\frac{-z_3}{1 - z_3} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1 - z_3)}{z_S - z_3}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S}{z_S - z_3} \right) + \right. \\
&\left. \frac{1}{1 - z_3} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S}{z_S - z_3} \right) \right] \tag{55}
\end{aligned}$$

In going from equation (54) to equation (55) we made use of the functional equation of Lauricella's hypergeometric function F_D , Proposition 2 (127), and Proposition 3 which are proved in Appendix A.

Now, for a light trajectory that encounters m turning points ($m \geq 1$) in the polar motion we have²:

$$\boxed{\pm \int_{\theta_S}^{\theta_{\min/\max}} \underbrace{\pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} \pm \int_{\theta_{\max/\min}}^{\theta_{\min/\max}} \cdots \pm \int_{\theta_{\max/\min}}^{\theta_O}}_{m-1 \text{ times}} =} \tag{56}$$

$$\begin{aligned}
&= \int_{z_S}^{z_m} + [1 - \text{sign}(\theta_S \circ \theta_{mS})] \int_0^{z_S} \\
&+ \int_{z_O}^{z_m} + [1 - \text{sign}(\theta_O \circ \theta_{mO})] \int_0^{z_O} \\
&+ 2(m-1) \int_0^{z_m} \tag{57}
\end{aligned}$$

where:

$$\boxed{\theta_{mO} := \text{Arccos}(\text{sign}(y_i)\sqrt{z_m}) = \text{Arccos}(\text{sign}(\beta_i)\sqrt{z_m}),} \tag{58}$$

²Recall the constraints of section 5.

y_i is the possible position of the image and:

$$\theta_{mS} := \begin{cases} \theta_{mO}, & m \text{ odd} \\ \pi - \theta_{mO}, & m \text{ even} \end{cases} \quad (59)$$

Thus we have that :

$$\begin{aligned} A_2(x_i, y_i, x_S, y_S, m) &= 2(m-1) \times \left[\frac{\Phi}{|a|} \frac{1}{\sqrt[2]{(z_m - z_3)}} \frac{1}{(1 - z_m)} \frac{\pi}{2} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{-z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) \right] \\ &+ \frac{\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_S)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1 - z_m)} \times \\ &F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S - z_m}{1 - z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \\ &+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{\Phi}{|a|} \frac{z_S}{z_m} \frac{z_S - z_m}{1 - z_S} \frac{1}{\sqrt[2]{z_S(z_S - z_m)(z_3 - z_S)}} \times \\ &F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1 - z_m)}{z_m(1 - z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) + \\ &+ \frac{\Phi}{2|a|} \sqrt[2]{\frac{(z_m - z_O)}{z_m}} \frac{1}{\sqrt[2]{z_m - z_3}} \frac{2}{(1 - z_m)} \times \\ &F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O - z_m}{1 - z_m}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3} \right) \\ &[1 - \text{sign}(\theta_O \circ \theta_{mO})] \frac{\Phi}{|a|} \frac{z_O}{z_m} \frac{z_O - z_m}{1 - z_O} \frac{1}{\sqrt[2]{z_O(z_O - z_m)(z_3 - z_O)}} \times \\ &F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_O(1 - z_m)}{z_m(1 - z_O)}, \frac{z_O}{z_m}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)} \right) \end{aligned} \quad (60)$$

We now calculate in closed form the angular term $A_1(x_i, y_i, x_S, y_S, m)$ which appears in equations (21),(4).

Indeed, the angular integrals of the form $\pm \int_{\theta_S}^{\theta_{\min}/\max}$, in equation (4), are computed in closed-analytic form in terms of Appell's generalized hypergeometric function of two variables as follows:

$$\begin{aligned} \pm \int_{\theta_S}^{\theta_{\min}/\max} \frac{d\theta}{\sqrt[2]{\Theta}} &= \frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_S)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1 - \text{sign}(\theta_s \circ \theta_{mS})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\ &F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) \end{aligned} \quad (61)$$

Also the integral (40) is calculated for $z_j = 0$ in terms of ordinary Gauß's hypergeometric function:

$$2(m-1) \int_0^{z_m} \frac{dz}{\sqrt[2]{z(z_m-z)(z-z_3)}} \quad (62)$$

$$= \frac{2(m-1)}{2|a|} \sqrt[2]{\frac{z_m}{z_m(z_m-z_3)}} \pi F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m-z_3}\right) \quad (63)$$

Thus we obtain,

$$\begin{aligned} A_1(x_i, y_i, x_S, y_S, m) &= 2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m-z_3)}} \pi F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m-z_3}\right) + \\ &\frac{1}{2|a|} \frac{\sqrt[2]{(z_m-z_S)}}{\sqrt[2]{z_m(z_m-z_3)}} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3}\right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt[2]{z_m-z_3}} \times \\ &F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m-z_3}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}\right) + \\ &\frac{1}{2|a|} \frac{\sqrt[2]{(z_m-z_O)}}{\sqrt[2]{z_m(z_m-z_3)}} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m-z_O}{z_m}, \frac{z_m-z_O}{z_m-z_3}\right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1 - \text{sign}(\theta_O \circ \theta_{mO})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}}}{\sqrt[2]{z_m-z_3}} \times \\ &F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m-z_3}, \frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}, \frac{z_O(z_m-z_3)}{z_m(z_O-z_3)}\right) \end{aligned} \quad (64)$$

For $m = 0$ i.e. for no turning points in the polar coordinate the exact solutions for the angular integrals in equation (4), (5) become

$$\begin{aligned}
A_1(x_i, y_i, x_S, y_S) &= \pm \int_{\theta_S}^{\theta_O} = \int_{z_1}^{z_2} + (1 - \text{sign}(\theta_S \circ \theta_O)) \int_0^{z_1} \\
&= \int_{z_1}^{z_m} - \int_{z_2}^{z_m} + (1 - \text{sign}(\theta_S \circ \theta_O)) \int_0^{z_1} \\
&= \frac{1}{2|a|} \frac{\sqrt[2]{z_m - z_1}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_1}{z_m}, \frac{z_m - z_1}{z_m - z_3} \right) 2 \\
&\quad - \frac{1}{2|a|} \frac{\sqrt[2]{z_m - z_2}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_2}{z_m}, \frac{z_m - z_2}{z_m - z_3} \right) 2 \\
&\quad + [1 - \text{sign}(\theta_S \circ \theta_O)] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_1(z_m - z_3)}{z_m(z_1 - z_3)}}}{\sqrt[2]{z_m - z_3}} \\
&\quad \times F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3}, \frac{z_1(z_m - z_3)}{z_m(z_1 - z_3)}, \frac{z_1(z_m - z_3)}{z_m(z_1 - z_3)} \right)
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
A_2(x_i, y_i, x_S, y_S) &= \frac{\Phi}{2|a|} \sqrt{\frac{(z_m - z_1)}{z_m}} \frac{1}{\sqrt{z_m - z_3}} \frac{2}{1 - z_m} \\
&\quad \times F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_1 - z_m}{1 - z_m}, \frac{z_m - z_1}{z_m}, \frac{z_m - z_1}{z_m - z_3} \right) \\
&\quad - \left[\frac{\Phi}{2|a|} \sqrt{\frac{(z_m - z_2)}{z_m}} \frac{1}{\sqrt{z_m - z_3}} \frac{2}{1 - z_m} \right. \\
&\quad \times F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_2 - z_m}{1 - z_m}, \frac{z_m - z_2}{z_m}, \frac{z_m - z_2}{z_m - z_3} \right) \Big] \\
&\quad + [1 - \text{sign}(\theta_S \circ \theta_O)] \frac{\Phi}{|a|} \sqrt{\frac{z_1}{z_m}} \sqrt{\frac{z_m - z_3}{z_1 - z_3}} \frac{1}{\sqrt{z_m - z_3}} \\
&\quad \times F_D \left(\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{z_1(1 - z_3)}{z_1 - z_3}, \frac{z_1}{z_m} \frac{z_m - z_3}{z_1 - z_3}, \frac{z_1}{z_1 - z_3} \right)
\end{aligned} \tag{66}$$

where $z_1 := \min(z_S, z_O)$, $z_2 := \max(z_S, z_O)$.

Equations (64), (60) for $m \geq 1$ turning points and (65), (66) for $m = 0$ turning points constitute our exact results for the angular integrals which appear in (21) for the case of vanishing cosmological constant Λ . It is time to turn our attention to the exact computation of the radial integrals which appear in the lens-equations of the Kerr black hole.

7 Closed form solution for the radial integrals.

We now perform the radial integration assuming $\Lambda = 0$.

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form:

$$\boxed{\int_{\alpha}^r \rightarrow -\int_{r_S}^{\alpha} + \int_{\alpha}^{r_O} \simeq 2 \int_{\alpha}^{\infty}} \quad (67)$$

For instance, in the calculation of the azimuthal coordinate (5) the following radial integral is involved:

$$\int_{\alpha}^{\infty} \frac{a}{\Delta} [(r^2 + a^2) - a\Phi] \frac{dr}{\sqrt{R}} \quad (68)$$

where $\Delta := r^2 + a^2 - 2GM \frac{r}{c^2}$. In order to calculate the contribution to the deflection angle from the radial term we need to integrate the above equation from the [distance of closest approach](#) (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic polynomial R (eqn (6) for $\Lambda = 0$) by $\alpha, \beta, \gamma, \delta : \alpha > \beta > \gamma > \delta$. We manipulate first the terms:

$$\boxed{\int_{\alpha}^{\infty} \frac{a}{\Delta} \frac{(r^2 + a^2)}{\sqrt{R}} dr = \int_{\alpha}^{\infty} \frac{adr}{\sqrt{R}} \left[1 + \underbrace{\frac{\frac{2GM}{c^2}r}{r^2 + a^2 - \frac{2GM}{c^2}r}}_{\Delta} \right] = \int_{\alpha}^{\infty} \frac{adr}{\sqrt{R}} + \int_{\alpha}^{\infty} \frac{a \frac{2GM}{c^2}r}{\Delta \sqrt{R}} dr} \quad (69)$$

It is enough to proceed with the term ³:

$$\int_{\alpha}^{\infty} \frac{a \frac{2GM}{c^2}r - a^2\Phi}{\Delta \sqrt{R}} dr \quad (70)$$

Expressing the roots of Δ as r_+, r_- , which are the radii of the event horizon and the inner or Cauchy horizon respectively, and using partial fractions we derive the expression:

$$\begin{aligned} \int_{\alpha}^{\infty} \frac{a \frac{2GM}{c^2}r - a^2\Phi}{\Delta \sqrt{R}} dr &= \int_{\alpha}^{\infty} \frac{A_+^{go}}{(r - r_+) \sqrt{R}} dr + \int_{\alpha}^{\infty} \frac{A_-^{go}}{(r - r_-) \sqrt{R}} dr \\ &= \int_{\alpha}^{\infty} \frac{A_+^{go}}{(r - r_+) \sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} dr \\ &\quad + \int_{\alpha}^{\infty} \frac{A_-^{go}}{(r - r_-) \sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} dr \end{aligned} \quad (71)$$

³The radial term $2 \int_{\alpha}^{\infty} \frac{adr}{\sqrt{R}}$ is cancelled from the angular term: $-\int \frac{a d\theta}{\sqrt{\Theta}}$.

where A_{\pm}^{go} are given by the equations

$$A_{\pm}^{go} = \pm \frac{(r_{\pm} a 2 \frac{GM}{c^2} - a^2 \Phi)}{r_+ - r_-} \quad (72)$$

For polar orbits $\Phi = 0$ and the coefficients in (72) reduce to those calculated in [6].

We organize all roots in ascending order of magnitude as follows⁴,

$$\alpha_{\mu} > \alpha_{\nu} > \alpha_i > \alpha_{\rho} \quad (73)$$

where $\alpha_{\mu} = \alpha_{\mu+1}$, $\alpha_{\nu} = \alpha_{\mu+2}$, $\alpha_{\rho} = \alpha_{\mu}$ and $\alpha_i = \alpha_{\mu-i}$, $i = 1, 2, 3$ and we have that $\alpha_{\mu-1} \geq \alpha_{\mu-2} > \alpha_{\mu-3}$. By applying the transformation

$$r = \frac{\omega z \alpha_{\mu+2} - \alpha_{\mu+1}}{\omega z - 1} \quad (74)$$

or equivalently

$$z = \left(\frac{\alpha_{\mu} - \alpha_{\mu+2}}{\alpha_{\mu} - \alpha_{\mu+1}} \right) \left(\frac{r - \alpha_{\mu+1}}{r - \alpha_{\mu+2}} \right) \quad (75)$$

where

$$\omega := \frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}} \quad (76)$$

we can bring our radial integrals into the familiar integral representation of Lauricella's F_D and Appell's hypergeometric function F_1 of three and two variables respectively. Indeed, we derive

$$\begin{aligned} \Delta \phi_{r_1}^{go} = & 2 \left[\int_0^{1/\omega} \frac{-A_+^{go} \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \frac{dz}{\sqrt[2]{z(1-z)}(1 - \kappa_+^2 z) \sqrt[2]{1 - \mu^2 z}} \right. \\ & + \int_0^{1/\omega} \frac{A_+^{go} \omega^2 (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \frac{z dz}{\sqrt[2]{z(1-z)}(1 - \kappa_+^2 z) \sqrt[2]{1 - \mu^2 z}} \\ & + \int_0^{1/\omega} \frac{-A_-^{go} \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} \frac{dz}{\sqrt[2]{z(1-z)}(1 - \kappa_-^2 z) \sqrt[2]{1 - \mu^2 z}} \\ & \left. + \int_0^{1/\omega} \frac{A_-^{go} \omega^2 (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} \frac{z dz}{\sqrt[2]{z(1-z)}(1 - \kappa_-^2 z) \sqrt[2]{1 - \mu^2 z}} \right] \quad (77) \end{aligned}$$

where the moduli κ_{\pm}^2, μ^2 are

$$\kappa_{\pm}^2 = \left(\frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}} \right) \left(\frac{\alpha_{\mu+2} - \alpha_{\mu-1}^{\pm}}{\alpha_{\mu+1} - \alpha_{\mu-1}^{\pm}} \right), \quad \mu^2 = \left(\frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}} \right) \left(\frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} \right) \quad (78)$$

⁴We have the correspondence $\alpha_{\mu+1} = \alpha, \alpha_{\mu+2} = \beta, \alpha_{\mu-1} = r_+ = \alpha_{\mu-2}, \alpha_{\mu-3} = \gamma, \alpha_{\mu} = \delta$.

Also

$$H^\pm = \sqrt[2]{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})(\alpha_{\mu+1} - \alpha_{\mu-1}^\pm) \sqrt[2]{\alpha_{\mu+1} - \alpha_\mu} \sqrt[2]{\alpha_{\mu+1} - \alpha_{\mu-3}} \quad (79)$$

and $\alpha_{\mu-1}^\pm = r_\pm$. By defining a new variable $z' := \omega z$ we can express the contribution $\Delta\phi_{r_1}^{go}$, to the deflection angle, from the above radial terms in terms of Lauricella's hypergeometric function F_D

$$\begin{aligned} \Delta\phi_{r_1}^{go} = & 2 \left[\frac{-2A_+^{go} \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+^{\prime 2}, \mu^{\prime 2} \right) \right. \\ & + \frac{A_+^{go} \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} F_D \left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_+^{\prime 2}, \mu^{\prime 2} \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \\ & + \frac{-2A_-^{go} \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_-^{\prime 2}, \mu^{\prime 2} \right) \\ & \left. + \frac{A_-^{go} \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} F_D \left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_-^{\prime 2}, \mu^{\prime 2} \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right] \quad (80) \end{aligned}$$

where the variables of the function F_D are given in terms of the roots of the quartic and the radii of the event and Cauchy horizons by the expressions

$$\begin{aligned} \frac{1}{\omega} &= \frac{\alpha_\mu - \alpha_{\mu+2}}{\alpha_\mu - \alpha_{\mu+1}} = \frac{\delta - \beta}{\delta - \alpha} \\ \kappa_\pm^{\prime 2} &= \frac{\alpha_{\mu+2} - \alpha_{\mu-1}^\pm}{\alpha_{\mu+1} - \alpha_{\mu-1}^\pm} = \frac{\beta - r_\pm}{\alpha - r_\pm} \\ \mu^{\prime 2} &= \frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} = \frac{\beta - \gamma}{\alpha - \gamma} \end{aligned} \quad (81)$$

An equivalent expression is as follows

$$\begin{aligned}
\Delta\phi_{r_1}^{go} &= 2 \left[\frac{-2A_+^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+^{\prime 2}, \mu^{\prime 2} \right) \right. \\
&\quad + \frac{A_+^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^+} \left(-\frac{1}{\kappa_+^{\prime 2}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^{\prime 2} \right) 2 \right. \\
&\quad \left. \left. + \frac{1}{\kappa_+^{\prime 2}} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_+^{\prime 2}, \mu^{\prime 2} \right) 2 \right) \right. \\
&\quad + \frac{-2A_-^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_-^{\prime 2}, \mu^{\prime 2} \right) \\
&\quad + \frac{A_-^{go}\sqrt{\omega}(\alpha_{\mu+1}-\alpha_{\mu+2})}{H^-} \left(-\frac{1}{\kappa_-^{\prime 2}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^{\prime 2} \right) 2 \right. \\
&\quad \left. \left. + \frac{1}{\kappa_-^{\prime 2}} F_D \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_-^{\prime 2}, \mu^{\prime 2} \right) 2 \right) \right] \\
&\equiv R_2(x_i, y_i)
\end{aligned} \tag{82}$$

In going from (80) to (82) we used the identity proven in [6], eqn.(52) in [6].

Finally, the term $\int_\alpha^\infty \frac{dr}{\sqrt{R}}$, is calculated in closed form in terms of Appell's first hypergeometric function of two-variables :

$$\begin{aligned}
\int_\alpha^\infty \frac{dr}{\sqrt{R}} &= \frac{(\alpha_{\mu+1}-\alpha_{\mu+2})}{\sqrt[3]{(\alpha_{\mu+2}-\alpha_{\mu+1})^2(\alpha_{\mu-1}-\alpha_{\mu+1})(\alpha_\mu-\alpha_{\mu+1})}} \times \\
&\quad \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \lambda_1^2, \frac{1}{\omega} \right) \\
&= \frac{1}{\sqrt{(\alpha-\gamma)(\alpha-\delta)}} \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta-\gamma}{\alpha-\gamma}, \frac{\delta-\beta}{\delta-\alpha} \right) \tag{83}
\end{aligned}$$

We exploit further the lens-equations (21). Indeed:

$$R_1(x_i, y_i) - 2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m-z_3)}} \pi F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m-z_3} \right) + \dots = \int^{\xi_S} \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} \tag{84}$$

Inverting:

$$\xi_S = \wp \left(\frac{2(83)}{1} - 2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m-z_3)}} \pi F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m-z_3} \right) + \dots + \epsilon \right) \tag{85}$$

while:

$$-\phi_S = R_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m) \tag{86}$$

where $\wp(z)$ denotes the Weierstraß elliptic function (which is also a meromorphic Jacobi modular form of weight 2) and the Weierstraß invariants are given in terms of the initial conditions by:

$$g_2 = \frac{1}{12}(\alpha + \beta)^2 - \mathcal{Q}\frac{\alpha}{4}, \quad (87)$$

$$g_3 = \frac{1}{216}(\alpha + \beta)^3 - \mathcal{Q}\frac{\alpha^2}{48} - \mathcal{Q}\frac{\alpha\beta}{48} \quad (88)$$

Also $\alpha := -a^2$, $\beta := \mathcal{Q} + \Phi^2$, $z_S = -\frac{\xi_S + \frac{\alpha + \beta}{2}}{-\alpha/4}$ and ϵ is a constant of integration.

To recapitulate our exact solutions of the lens equations (21) are given by:

$$\begin{aligned} 2 \int_{\alpha}^{\infty} \frac{1}{\sqrt{R}} dr &= A_1(x_i, y_i, x_S, y_S, m) \Leftrightarrow \frac{2}{\sqrt{(\alpha - \gamma)(\alpha - \delta)}} \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta}{\delta - \alpha} \right) = \\ &2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} \pi F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) + \\ &\frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_S)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\ &F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) + \\ &\frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_O)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\ &+ [1 - \text{sign}(\theta_O \circ \theta_{mO})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\ &F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)}, \frac{z_O(z_m - z_3)}{z_m(z_O - z_3)} \right), \end{aligned} \quad (89)$$

$$- \phi_S = R_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m) \quad (90)$$

and equation (85).

In the subsequent sections we shall apply our exact solutions for the lens equations in the Kerr geometry expressed by (89), (85), (90) to particular cases which include: a) an equatorial observer: $\theta_O = \pi/2 \Rightarrow z_O = 0$ b) a generic observer located at $\theta_O = \pi/3 \Rightarrow z_O = \frac{1}{4}$.

8 Positions of images, source and resulting magnifications for an equatorial observer in a Kerr black hole.

In this case ($\theta_O = \pi/2$), equations (15),(16), become:

$$\begin{aligned}\Phi &\simeq -\alpha_i \sin \theta_O = -\alpha_i \\ \mathcal{Q} &\simeq \beta_i^2 + (\alpha_i^2 - a^2) \cos^2 \theta_O = \beta_i^2\end{aligned}\tag{91}$$

Thus the length of the vector on the observer's image plane equals to:

$$\sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\Phi^2 + \mathcal{Q}}\tag{92}$$

Furthermore, we derive the equations:

$$\boxed{x_S := \frac{\alpha_S}{r_O} = \frac{r_S \sin \theta_S \sin \phi_S}{r_O - r_S \sin \theta_S \cos \phi_S}}\tag{93}$$

$$\boxed{y_S := \frac{\beta_S}{r_O} = \frac{-r_S \cos \theta_S}{r_O - r_S \sin \theta_S \cos \phi_S}}\tag{94}$$

or equivalently:

$$\frac{\alpha_S}{\beta_S} = -\tan \theta_S \sin \phi_S\tag{95}$$

8.1 Solution of the lens equation and the computation of $\theta_S, \phi_S, \alpha_i, \beta_i$.

We now describe how we solve the lens equations (21) using the properties of the Weierstraß Jacobi modular form $\wp(z)$ equation (85) and the computation of the radial and angular integrals in terms of Appell-Lauricella hypergeometric functions equations (83),(82),(64),(60) respectively.

For a choice of initial conditions a, Φ, \mathcal{Q} we determine values for the observer image plane coordinates α_i, β_i , see equation (91). Subsequently we determine

the value of z_S and therefore of θ_S that satisfies the equation ⁵ :

$$\begin{aligned}
2 \int_{\alpha}^{\infty} \frac{1}{\sqrt{R}} dr &= A_1(x_i, y_i, x_S, y_S, m) \Leftrightarrow \frac{2}{\sqrt{(\alpha - \gamma)(\alpha - \delta)}} \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta}{\delta - \alpha} \right) = \\
&2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} \pi F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) + \\
&\frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_S)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\
&+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\
&F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) + \\
&\frac{1}{2|a|} \frac{\sqrt[2]{1}}{\sqrt[2]{(z_m - z_3)}} F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) \pi
\end{aligned} \tag{96}$$

using our exact solution for z_S in terms of the Weierstraß elliptic function equation (85). For this we need to know at which regions of the fundamental period parallelogram the Weierstraß function takes real and negative values. Indeed, the function of Weierstraß takes the required values at the points: $x = \frac{\omega}{l} + \omega', l \in \mathbb{R}$ of the fundamental region $(\wp(\frac{\omega}{l} + \omega'; g_2, g_3) \in \mathbb{R}^-)$. Thus as the parameter l varies we determine the value of z_S that satisfies equations (85), (96). The quantities ω, ω' denote the Weierstraß half-periods. In the case under investigation ω is a real half-period while ω' is pure imaginary. For positive discriminant $\Delta_c = g_2^3 - 27g_3^2$, all three roots e_1, e_2, e_3 of $4z^3 - g_2z - g_3$ are real and if the e_i are ordered so that $e_1 > e_2 > e_3$ we can choose the half-periods as

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}, \quad \omega' = i \int_{-\infty}^{e_3} \frac{dt}{\sqrt{-4t^3 + g_2t + g_3}} \tag{97}$$

The period ratio τ is defined by $\tau = \omega'/\omega$. An alternative expression for the real half-period ω of the Weierstraß elliptic function is given by the hypergeometric function of Gauß⁶:

$$\omega = \frac{1}{\sqrt{e_1 - e_3}} \frac{\pi}{2} F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{e_2 - e_3}{e_1 - e_3} \right) \tag{98}$$

Having determined θ_S by the procedure we just described we determine the

⁵Which is the closed form solution of the radial and angular integrals of the first of lens equations in eqn (21).

⁶The three roots are given in terms of the first integrals of motion by the expressions: $e_1 = \frac{1}{24}(-a^2 + Q + \Phi^2 + 3\sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2})$, $e_2 = \frac{1}{12}(a^2 - Q - \Phi^2)$, $e_3 = \frac{1}{24}(-a^2 + Q + \Phi^2 - 3\sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2})$

	$a = 0.6, \mathcal{Q} = 24.64563, \Phi = -2.719110$	$a = 0.6, \mathcal{Q} = 0.128, \Phi = 3.839$
$\alpha_i \left(\frac{GM}{c^2} \right)$	2.719110	-3.839
$\beta_i \left(\frac{GM}{c^2} \right)$	-4.9644365239	0.357770876399
$x_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	1.359555	-1.9195
$y_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	-2.48221826	0.178885
m	3	3
z_S	0.3161007914992452	0.0026145818604
θ_S	55.79°	87.069°
$\Delta\phi(\text{rad})$	-11.086	7.09441
ϕ_S	95.1794°	133.52°
ω	0.5545341990201503500	0.824718843878947
ω'	1.3278669366032567973i	2.9400828459149726i

Table 1: Solution of the lens equations in Kerr geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/2, \phi_O = 0$. The number of turning points in the polar variable is three. The values for the Kerr parameter and the impact factor Φ are in units of $\frac{GM}{c^2}$ while those of Carter's constant \mathcal{Q} are in units of $\frac{G^2 M^2}{c^4}$.

azimuthial position of the source ϕ_S by the second equation of (21):

$$-\phi_S = R_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m) \quad (99)$$

Let us give an example. For the choice $\mathcal{Q} = 24.64563 \frac{G^2 M^2}{c^4}, \Phi = -2.719110 \frac{GM}{c^2}, a = 0.6 \frac{GM}{c^2}$ we determine $z_S = 0.3161007914992452, m = 3$ and $\Delta\phi = -11.086, \phi_S = 95.1794^\circ$. Keeping fixed the value of the Kerr parameter we solved the lens equations for different values of Carter's constant \mathcal{Q} and impact factor Φ . We exhibit our results in table 1⁷.

Let us at this point, present a solution with a higher value for the Kerr parameter in Table 2.

The positions of the images of Tables 1 and 2 on the observer's image plane are displayed in fig.2 and fig.3 respectively. In the same figures the boundary of the shadow of the spinning black hole is also displayed.

8.2 Closed form calculation for the magnifications.

We outline in this subsection, the closed-form calculation, of the resulting magnification factors. It turns out that the derivatives involved in the expression for the magnification are elegantly computed using the beautiful property of

⁷ Assuming that the galactic centre region, SgrA*, is a Kerr black hole with mass: $M_{\text{BH}} = 10^6 M_\odot$ and a distance from the observer to the galactic centre: $r_O = 8\text{Kpc}$, the second solution in Table 1 will require an angular resolution of $19.3102\mu\text{arcs}$. This is in the range of experimental accuracy for both the TMT and GRAVITY experiments.

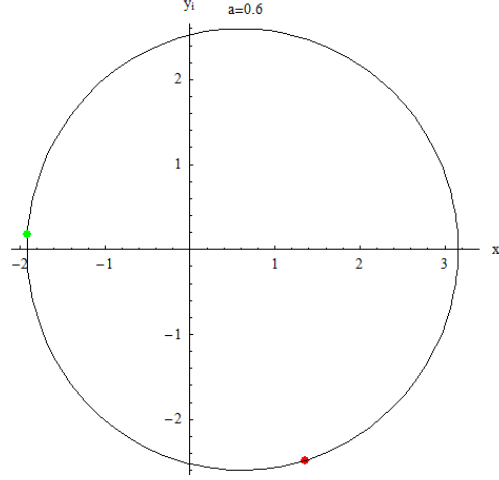


Figure 2: The two images of Table 1 on the observer's image plane. The value of the Kerr parameter is $a = 0.6 \frac{GM}{c^2}$, while the observer is located at $\theta_O = \pi/2$. With red is the image solution, first column of Table 1 and with green the image solution, second column of table 1.

	$a = 0.9939, \mathcal{Q} = 27.0220588123, \Phi = -2.29885534$
$\alpha_i \left(\frac{GM}{c^2} \right)$	2.29885534
$\beta_i \left(\frac{GM}{c^2} \right)$	5.198274599547431
$x_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	1.14942767
$y_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	2.5991372997737154
m	3
z_S	0.01378435185109
θ_S	83.2575°
$\Delta\phi(\text{rad})$	-11.243
ϕ_S	104.177°
ω	0.5505433970950226
ω'	1.1288708298860726 i

Table 2: Solution of the lens equations in Kerr geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/2, \phi_O = 0$ for a high value for the spin of the black hole. The number of turning points in the polar variable is three. The values for the Kerr parameter and the impact factor Φ are in units of $\frac{GM}{c^2}$ while those of Carter's constant \mathcal{Q} are in units of $\frac{G^2 M^2}{c^4}$.

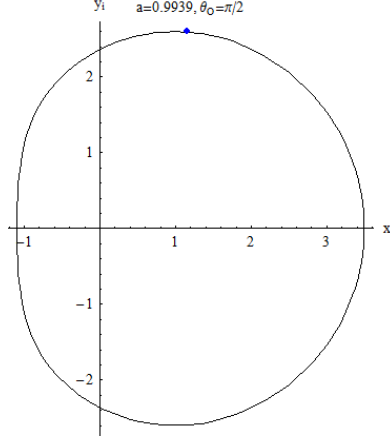


Figure 3: The lens solution of Table 2 as it will be detected on the observer image plane by an equatorial observer. The boundary of the shadow of the black hole is also exhibited.

the hypergeometric functions: namely, that the derivatives of the hypergeometric functions of Appell-Lauricella are again hypergeometric functions of the same type with a different set of parameters. It is a powerful property of our formalism which we exploit to the full in what follows.

$$\begin{aligned}
\frac{\partial(55)}{\partial x_S} &= \frac{\partial(55)}{\partial z_S} \frac{\partial z_S}{\partial x_S}, \\
\frac{\partial(55)}{\partial z_S} &= \frac{\Phi}{2|a|} \frac{1}{z_m} \frac{1}{(1-z_m)} \frac{1}{\sqrt{z_m-z_3}} \left(\frac{z_m-z_S}{z_m} \right)^{-1/2} \times \\
&F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S-z_m}{1-z_m}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) + \\
&\left(\frac{-\Phi}{2|a|} \sqrt{\frac{(z_m-z_S)}{z_m}} \frac{1}{\sqrt{z_m-z_3}} \frac{2}{(1-z_m)} \right) \times \left\{ \right. \\
&F_D \left(\frac{3}{2}, 2, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_S-z_m}{1-z_m}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) \frac{1}{1-z_m} + \\
&F_D \left(\frac{3}{2}, 1, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_S-z_m}{1-z_m}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) \frac{-1}{z_m} + \\
&F_D \left(\frac{3}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z_S-z_m}{1-z_m}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) \frac{-1}{z_m-z_3} \left. \right\} + \\
&(1 - \text{sign}(\theta_S \circ \theta_{ms})) (-1) \left[\frac{1}{z_m} \frac{z_S-z_m}{1-z_S} \frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}} + \right. \\
&\left. \frac{z_S}{z_m} \frac{z_3(z_m-3z_Sz_m+2z_S^2) - z_S(z_m(2-4z_S) + z_S(-1+3z_S))}{2(1-z_S)^2 z_S(z_3-z_S) \sqrt{z_S(z_S-z_m)(z_3-z_S)}} \right] \times \\
&F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) + \\
&\frac{z_S}{z_m} \frac{z_S-z_m}{(1-z_S)} \frac{1}{\sqrt{z_S(z_S-z_m)(z_3-z_S)}} \left\{ \right. \\
&F_D \left(2, 2, -\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) \frac{1-z_m}{z_m(1-z_S)^2} + \\
&F_D \left(2, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) \frac{1}{z_m} + \\
&F_D \left(2, 1, \frac{-1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z_S(1-z_m)}{z_m(1-z_S)}, \frac{z_S}{z_m}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) \left(\frac{-z_3(z_m-z_3)}{z_m(z_S-z_3)^2} \right) \left. \right\} \Big] \\
&\hspace{15em} (100)
\end{aligned}$$

Thus,

$$\frac{\partial(55)}{\partial x_S} = (100) \times \left(-2 \cos \theta_S \sin \theta_S \frac{r_S^2 \sin \theta_S \cos \theta_S \sin \phi_S}{(r_O - r_S \sin \theta_S \cos \phi_S)^2} \right) \quad (101)$$

Now we calculate the term: $\frac{\partial(61)}{\partial z_S}$. Indeed, calculating the derivatives w.r.t. z_S we derive the expression:

$$\begin{aligned}
\frac{\partial(61)}{\partial z_S} = & \frac{1}{2|a|} \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)} \left(-\frac{1}{2\sqrt{z_m(z_m-z_3)}\sqrt{z_m-z_S}} \right) F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) + \\
& \frac{1}{2|a|} \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)} \sqrt{\frac{(z_m-z_S)}{z_m(z_m-z_3)}} \times \left[F_1 \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) \left(\frac{-1}{z_m} \right) + \right. \\
& F_1 \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z_m-z_S}{z_m}, \frac{z_m-z_S}{z_m-z_3} \right) \left(\frac{-1}{z_m-z_3} \right) \left. \right] + \\
& [1 - \text{sign}(\theta_S \circ \theta_{ms})] \left[\frac{1}{2|a|} \left(\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right)^{-\frac{1}{2}} \left\{ \frac{(-z_3)\sqrt{z_m-z_3}}{z_m(z_S-z_3)^2} \right\} \times \right. \\
& F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m-z_3} \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) + \\
& \frac{1}{|a|} \frac{\sqrt{\frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}}}{\sqrt{z_m-z_3}} \times \left[F_1 \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_m}{z_m-z_3} \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) \left(\frac{-z_3}{(z_S-z_3)^2} \right) + \right. \\
& F_1 \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z_m}{z_m-z_3} \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)}, \frac{z_S(z_m-z_3)}{z_m(z_S-z_3)} \right) \left(\frac{(-z_3)(z_m-z_3)}{z_m(z_S-z_3)^2} \right) \left. \right] \left. \right]
\end{aligned} \tag{102}$$

Now:

$$\alpha_1 = \frac{\partial A_1}{\partial x_S} = (102) \times \frac{\partial z_S}{\partial x_S} = (102) \times \left(-2 \cos \theta_S \sin \theta_S \times \frac{\frac{r_S^2 \sin \theta_S \cos \theta_S \sin \phi_S}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} \right) \tag{103}$$

and

$$\alpha_2 = \frac{\partial A_1}{\partial y_S} = (102) \times \frac{\partial z_S}{\partial y_S} = (102) \times \left(-2 \cos \theta_S \sin \theta_S \times \frac{\frac{-[r_O r_S \sin \theta_S \cos \phi_S - r_S^2 \sin^2 \theta_S]}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} \right) \tag{104}$$

While for the α_3, α_4 terms which contribute to the expression for the magnification. equation (26), we derive the expressions:

$$\alpha_3 = -\frac{\partial \phi_S}{\partial x_S} - \frac{\partial A_2}{\partial x_S} = -\left(-\frac{\frac{(r_O r_S \sin \theta_S - r_S^2 \cos \phi_S)}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} \right) - (100) \times \left(\frac{\frac{r_S^2 \sin \theta_S \cos \theta_S \sin \phi_S}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} \right) \tag{105}$$

$$\alpha_4 = -\frac{\partial \phi_S}{\partial y_S} - \frac{\partial A_2}{\partial y_S} = -\frac{\frac{r_O r_S \cos \theta_S \sin \phi_S}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} - (100) \times \frac{\frac{-[r_O r_S \sin \theta_S \cos \phi_S - r_S^2 \sin^2 \theta_S]}{(r_O - r_S \sin \theta_S \cos \phi_S)^2}}{J_1} \quad (106)$$

where J_1 denotes the Jacobian:

$$J_1 = \frac{\partial(x_S, y_S)}{\partial(\theta_S, \phi_S)} \quad (107)$$

and

$$\begin{aligned} \frac{\partial \theta_S}{\partial x_S} &= \frac{(r_S^2 \sin \theta_S \cos \theta_S \sin \phi_S)/((r_O - r_S \sin \theta_S \cos \phi_S)^2)}{J_1} \\ \frac{\partial \theta_S}{\partial y_S} &= \frac{-[r_O r_S \sin \theta_S \cos \phi_S - r_S^2 \sin^2 \theta_S]/((r_O - r_S \sin \theta_S \cos \phi_S)^2)}{J_1} \\ \frac{\partial \phi_S}{\partial x_S} &= -\frac{(r_O r_S \sin \theta_S - r_S^2 \cos \phi_S)/((r_O - r_S \sin \theta_S \cos \phi_S)^2)}{J_1} \\ \frac{\partial \phi_S}{\partial y_S} &= \frac{r_O r_S \cos \theta_S \sin \phi_S/((r_O - r_S \sin \theta_S \cos \phi_S)^2)}{J_1} \end{aligned} \quad (108)$$

In producing the results exhibited in eqns (100),(102) in our calculations for the magnification factors we made use of the important identity of Appell's hypergeometric function F_1 :

$$\frac{\partial^{m+n} F_1(\alpha, \beta, \beta', \gamma, x, y)}{\partial x^m \partial y^n} = \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \times F_1(\alpha + m + n, \beta + m, \beta' + n, \gamma + m + n, x, y) \quad (109)$$

and its corresponding generalization for the fourth hypergeometric function of Lauricella. Similar calculations that we do not exhibit in this written account of our talk, lead to the derivation of the coefficients $\beta_1, \beta_2, \beta_3, \beta_4$ in terms of the generalized hypergeometric functions of Appell-Lauricella.

A phenomenological analysis of our exact solutions for the magnifications in Kerr spacetime will be a subject of a separate publication [15].

8.3 Source and image positions for an observer located at $\theta_O = \frac{\pi}{3}$.

In this case, the coordinates on the observers image plane are related to the first integrals of motions as follows:

$$\Phi = -\alpha_i \frac{\sqrt{3}}{2}, \quad Q = \beta_i^2 + \left(\frac{4\Phi^2}{3} - a^2 \right) \frac{1}{4}. \quad (110)$$

Furthermore, our solution for the first lens equation (89) takes the form:

$$\begin{aligned}
2 \int_{\alpha}^{\infty} \frac{1}{\sqrt{R}} dr &= A_1(x_i, y_i, x_S, y_S, m) \Leftrightarrow \frac{2}{\sqrt{(\alpha - \gamma)(\alpha - \delta)}} \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta}{\delta - \alpha} \right) = \\
&2(m-1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} \pi F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_m}{z_m - z_3} \right) + \\
&\frac{1}{2|a|} \frac{\sqrt[2]{(z_m - z_S)}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\
&+ [1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\
&F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)}, \frac{z_S(z_m - z_3)}{z_m(z_S - z_3)} \right) + \\
&\frac{1}{2|a|} \frac{\sqrt[2]{(z_m - \frac{1}{4})}}{\sqrt[2]{z_m(z_m - z_3)}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - \frac{1}{4}}{z_m}, \frac{z_m - \frac{1}{4}}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(3/2)} \\
&+ [1 - \text{sign}(\frac{\pi}{3} \circ \theta_{mO})] \frac{1}{|a|} \frac{\sqrt[2]{\frac{(1/4)(z_m - z_3)}{z_m((1/4) - z_3)}}}{\sqrt[2]{z_m - z_3}} \times \\
&F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \frac{(1/4)(z_m - z_3)}{z_m(1/4 - z_3)}, \frac{(1/4)(z_m - z_3)}{z_m((1/4) - z_3)} \right). \tag{111}
\end{aligned}$$

Let us give now examples of solutions for the images and source positions of equations (111),(85).

For the initial conditions $\mathcal{Q}=25.64563, a=0.9939, \Phi=-3.11$ we calculate the parameter z_S of the source latitude position to be: $z_S=0.09097820848$ ($\theta_S=72.4447^\circ$) for $m=3$. The position at the fundamental period parallelogram that provides the above value of z_S as a solution of (111),(85) for three turning points in the polar variable is located at: $\frac{\omega}{1.2995480690017123} + \omega'$, where the fundamental half-periods of the Weierstraß elliptic function $\wp(x, g_2, g_3)$ were calculated to be:

$$\omega = 0.52792338858688228, \quad \omega' = 1.119903617249492 \text{ i} \tag{112}$$

Also the azimuthial position of the source was calculated to be using (90) and the calculated value of z_S : $\phi_S = 2.67231589\text{rad} = 153.112^\circ = 551205''^8$.

The first solution is shown on the image plane of the observer, Fig.4. We observe that the solution lies close to the boundary of the shadow of the black hole.

⁸ $\Delta\phi = R_2(x_i, y_i) + A_2(x_i, y_i, x_s, y_s, m)$ was calculated to be: $\Delta\phi = -12.0971\text{rad}$ so that the photons perform more than one loop and a half around the black hole.

	$a = 0.9939, \mathcal{Q} = 25.64563, \Phi = -3.11$	$a = 0.52, \mathcal{Q} = 23.64563, \Phi = -2.85$
$\alpha_i \left(\frac{GM}{c^2} \right)$	3.591118674	3.29089
$\beta_i \left(\frac{GM}{c^2} \right)$	-4.7611506980	-4.58320084657
$x_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	1.79556	1.64545
$y_i \left(\frac{2}{r_O} \frac{GM}{c^2} \right)$	-2.38058	-2.2916004
m	3	3
z_S	0.09097820848	0.5980171072414
θ_S	72.4447°	39.3474°
$\Delta\phi(\text{rad})$	-12.0971	-11.8577
ϕ_S	153.112°	139.395°
ω	0.52792338858688228	0.5571026427501503
ω'	1.119903617249492 i	1.389041935594241i

Table 3: Solution of the lens equations in Kerr geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/3, \phi_O = 0$. The number of turning points in the polar variable is three. The values for the Kerr parameter and the impact factor are in units of $\frac{GM}{c^2}$.

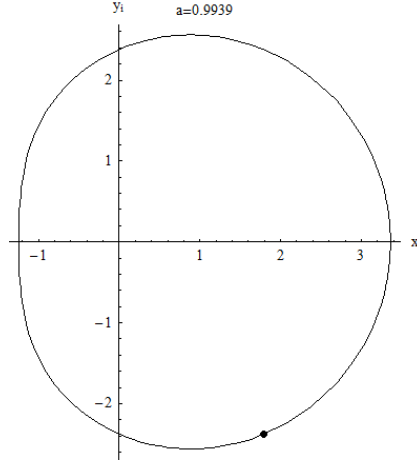


Figure 4: The solution, 1st column of Table 3 as it will be detected on the observer image plane by an observer at $\theta_O = \pi/3, \phi_O = 0$. The boundary of the shadow of the black hole is also exhibited.

9 Exact solution of the angular integrals in the presence of the cosmological constant Λ .

There has been a discussion in the literature as to whether or not the cosmological constant contributes to the gravitational lensing. However, the debate has been restricted to the Schwarzschild-de Sitter spacetime [16], [17],[18]. Let us discuss now the more general case of gravitational lensing in the Kerr-de Sitter spacetime.

The generalized solution for the angular integral (54) in the presence of Λ is given by:

$$\begin{aligned} \pm \int_{\theta_S}^{\theta_{\min/\max}} &= \frac{\Xi^2}{2|H|} \frac{z_m - z_S}{(1 - \eta z_m)} \frac{1}{\sqrt{z_m(z_m - z_S)(z_m - z_3)}} \times \left\{ \right. \\ &\frac{\Phi}{(1 - z_m)} F_D \left(\frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\eta(z_S - z_m)}{1 - \eta z_m}, \frac{z_S - z_m}{1 - z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} + \\ &\left. - a F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\eta(z_S - z_m)}{1 - \eta z_m}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \right\} \\ &(1 - \text{sign}(\theta_S \circ \theta_{mS})) \left[\frac{\Xi^2}{|H|} \frac{z_S}{z_m} \frac{z_m - z_S}{1 - \eta z_S} \frac{1}{\sqrt{z_S(z_S - z_m)(z_3 - z_S)}} \times \right. \\ &\left. \left\{ -a F_D \left(1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \lambda, \frac{z_S}{z_m}, \mu \right) + \frac{\Phi}{1 - z_S} F_D \left(1, 1, 1, -\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \lambda, \nu, \frac{z_S}{z_m}, \mu \right) \right\} \right] \end{aligned} \quad (113)$$

where:

$$\boxed{\eta := -\frac{a^2 \Lambda}{3}, \mu = \frac{z_S}{z_m} \frac{z_m - z_3}{z_S - z_3}, \lambda = \frac{z_S}{z_m} \left(\frac{1 - \eta z_m}{1 - \eta z_S} \right), \nu = \frac{z_S}{z_m} \left(\frac{1 - z_m}{1 - z_S} \right)} \quad (114)$$

Also the integrals $\pm \int_{\theta_{\min/\max}}^{\theta_{\max/\min}} = 2 \int_0^{z_m}$ contribute the term:

$$\boxed{2(m-1) \times \left\{ \begin{aligned} &\frac{\Xi^2 \Phi}{2|H|} \frac{z_m}{(1 - \eta z_m)(1 - z_m)} \frac{1}{\sqrt{z_m^2(z_m - z_3)}} \\ &\times F_D \left(\frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\eta(-z_m)}{1 - \eta z_m}, \frac{-z_m}{1 - z_m}, 1, \frac{z_m}{z_m - z_3} \right) 2 \\ &+ \frac{-\Xi^2 a}{2|H|} \frac{z_m}{\sqrt{z_m^2(z_m - z_3)}} \frac{1}{1 - \eta z_m} \\ &\times F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\eta(-z_m)}{1 - \eta z_m}, 1, \frac{z_m}{z_m - z_3} \right) 2 \end{aligned} \right\}} \quad (115)$$

Notice that for $\Lambda = 0$ this reduces to equation (53).

9.1 Closed-form solution for the radial integrals in the presence of the cosmological constant Λ .

Assume $\Lambda > 0$. We need to calculate radial integrals of the form:

$$\int \frac{a\Xi^2}{\Delta_r} ((r^2 + a^2) - a\Phi) \frac{dr}{\sqrt[2]{R}} \quad (116)$$

We use the technique of partial fractions from integral calculus:

$$\frac{a\Xi^2}{\Delta_r} ((r^2 + a^2) - a\Phi) = \frac{A^1}{r - r_\Lambda^+} + \frac{A^2}{r - r_\Lambda^-} + \frac{A^3}{r - r_+} + \frac{A^4}{r - r_-} \quad (117)$$

where $r_\Lambda^+, r_\Lambda^-, r_+, r_-$ are the four real roots of Δ_r .

For instance, for $r_O, r_S < r_\Lambda^+$ one of the integrals we need to calculate is:

$$\frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda + 3\Xi^2(1 + \frac{\Lambda}{3}(a - \Phi)^2))}} \int_\alpha^{r_\Lambda^+/2} \frac{A^1 dr}{(r - r_\Lambda^+) \sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} \quad (118)$$

Indeed, we compute in closed-form:

$$\begin{aligned} & \int_\alpha^{r_\Lambda^+/2} \frac{A^1 dr}{(r - r_\Lambda^+) \sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} \\ &= \frac{\rho_1}{\sqrt{\rho_1}} H_\Lambda^+ \times \\ & F_D \left(\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{\beta - \gamma}{\alpha - \gamma} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{\beta - \delta}{\alpha - \delta} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{r_\Lambda^+ - \beta}{r_\Lambda^+ - \alpha} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \end{aligned} \quad (119)$$

where

$$\rho_1 := \frac{r_\Lambda^+ - \beta}{r_\Lambda^+ - \alpha} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta} \quad (120)$$

Also the radial integral involved on the LHS in the “balance” lens equation (4) is computed exactly in terms of the hypergeometric function of Appell F_1 :

$$\begin{aligned} & \frac{1}{\sqrt{\frac{1}{3}(\mathcal{Q}\Lambda + 3\Xi^2(1 + \frac{\Lambda}{3}(a - \Phi)^2))}} \int_\alpha^{r_\Lambda^+/2} \frac{dr}{\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} \\ &= \frac{\rho_1}{\sqrt{\mathcal{E}}} \frac{1}{\sqrt{\omega(\gamma - \alpha)(\delta - \alpha)}} \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - \gamma}{\alpha - \gamma} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta}, \frac{\beta - \delta}{\alpha - \delta} \frac{r_\Lambda^+ - 2\alpha}{r_\Lambda^+ - 2\beta} \right) \end{aligned} \quad (121)$$

where $\mathcal{E} := \frac{1}{3}(\mathcal{Q}\Lambda + 3\Xi^2(1 + \frac{\Lambda}{3}(a - \Phi)^2)$, $\omega := \frac{r_{\Lambda}^+ - \alpha}{r_{\Lambda}^+ - \beta}$ and $\alpha, \beta, \gamma, \delta$ denote the roots of the quartic polynomial R in the presence of Λ eqn(6).

A complete phenomenological analysis of our exact solutions in the presence of the cosmological constant Λ will be a subject of a separate publication [15]. Nevertheless, it is evident from the closed form solutions we derived in this work that the cosmological constant *does* contribute to the gravitational bending of light.

10 Conclusions

In this work the precise analytic treatment of Kerr and Kerr-de Sitter black holes as gravitational lenses has been achieved. A full analytic strong-field calculation of the source, image positions and the resulting magnification factors has been performed. A full blend of important functions from Mathematical Analysis such as the Weierstraß elliptic function \wp and the generalized multivariable hypergeometric functions of Appell-Lauricella F_D were deployed in deriving the closed-form solution of the gravitational lens equations. From the exact solution of the radial and angular Abelian integrals which are involved in the lens equations we concluded the Λ does contribute to the gravitational bending of light. A full quantitative phenomenological analysis of gravitational lensing by a Kerr deflector in the presence of Λ is beyond the scope of this work and will appear elsewhere [15]. We provided examples of image-source configurations that solve the gravitational Kerr lens equations and exhibited their appearance on the observer's image plane as they will be detected by an equatorial observer ($\theta_O = \pi/2, \phi_O = 0$) and an observer located at $\theta_O = \pi/3, \phi_O = 0$, for various values of the Kerr parameter a , and the first integrals of motion Φ, \mathcal{Q} .

The theory produced in this work based on the exact solution of the null geodesic equations of motion in Kerr spacetime will have an important application to the Sgr A* galactic centre supermassive black hole [15]. It may serve the important goal of probing general relativity at the strong field regime through the phenomenon of gravitational bending of light induced by the spacetime curvature. It is complementary to other investigations which have the ambition to probe gravitation at the strong-field regime through the relativistic effects of periastron precession and frame-dragging [19].

There is a fruitful synergy of various fields of Science: general relativity, astronomy, cosmology and pure mathematics.

11 Acknowledgments

This is modified written account of the author's talk at NEB-14 Recent Developments in Gravity that took place at the University of Ioannina. The author would like to thank: L. Perivolaropoulos and P. Kanti for inviting him to deliver his talk to such an exciting conference. He is also obliged to C. E. Vayonakis

for discussions and comments on the manuscript. He warmly thanks his colleagues and his undergraduate students at the Physics department, University of Ioannina, for a stimulating academic environment. In addition, he thanks G. Kakarantzas for discussions and his friendship. Last but not least, he thanks his family for moral support during the early stages of this work.

A Transformation properties of Lauricella's hypergeometric function F_D .

In this appendix we prove useful transformation properties of Lauricella's hypergeometric function F_D . We first introduce the function and its integral representation:

Lauricella's 4th hypergeometric function of m-variables.

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) = \sum_{n_1, n_2, \dots, n_m=0}^{\infty} \frac{(\alpha)_{n_1+\dots+n_m} (\beta_1)_{n_1} \dots (\beta_m)_{n_m}}{(\gamma)_{n_1+\dots+n_m} (1)_{n_1} \dots (1)_{n_m}} z_1^{n_1} \dots z_m^{n_m} \quad (122)$$

where

$$\begin{aligned} \mathbf{z} &= (z_1, \dots, z_m), \\ \boldsymbol{\beta} &= (\beta_1, \dots, \beta_m). \end{aligned} \quad (123)$$

The Pochhammer symbol $(\alpha)_m = (\alpha, m)$ is defined by

$$(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } m = 0 \\ \alpha(\alpha + 1) \dots (\alpha + m - 1) & \text{if } m = 1, 2, 3 \end{cases} \quad (124)$$

The series admits the following integral representation:

$$F_D(\alpha, \boldsymbol{\beta}, \gamma, \mathbf{z}) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-z_1 t)^{-\beta_1} \dots (1-z_m t)^{-\beta_m} dt \quad (125)$$

which is valid for $\boxed{\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\gamma - \alpha) > 0}$. It *converges absolutely* inside the m-dimensional cuboid:

$$|z_j| < 1, (j = 1, \dots, m). \quad (126)$$

Proposition 2 *The following holds:*

$$\begin{aligned} F_D(\alpha, \beta, \beta', \beta'', \gamma, x, y, z) &= (1-z)^{-\alpha} \times \\ &\quad F_D\left(\alpha, \beta, \beta', \gamma - \beta - \beta' - \beta'', \gamma, \frac{z-x}{z-1}, \frac{z-y}{z-1}, \frac{z}{z-1}\right) \end{aligned} \quad (127)$$

Proof. Applying the tranformation in equation (48):

$$u = \frac{\nu}{1 - z + \nu z} = \frac{\nu}{(1 - z) \left[1 - \frac{\nu z}{z-1} \right]} \quad (128)$$

we get:

$$\begin{aligned} 1 - u &= \frac{1 - \nu}{1 - \nu \frac{z}{z-1}}, \quad (1 - ux)^{-\beta} = \left(\frac{[1 - \frac{\nu(z-x)}{z-1}]}{[1 - \frac{\nu z}{z-1}]} \right)^{-\beta} \\ (1 - u y)^{-\beta'} &= \left(\frac{[1 - \frac{\nu(z-y)}{z-1}]}{[1 - \frac{\nu z}{z-1}]} \right)^{-\beta'}, \quad (1 - uz)^{-\beta''} = \left(\frac{1}{[1 - \frac{\nu z}{z-1}]} \right)^{-\beta''} \end{aligned} \quad (129)$$

Thus

$$\begin{aligned} IRF_D &= (1 - z)^{-\alpha} \times \\ &\int_0^1 d\nu \nu^{\alpha-1} (1 - \nu)^{\gamma-\alpha-1} (1 - \nu \frac{z-x}{z-1})^{-\beta} (1 - \nu \frac{z-y}{z-1})^{-\beta'} (1 - \nu \frac{z}{z-1})^{-(\gamma-\beta-\beta'-\beta'')} \end{aligned} \quad (130)$$

and proposition follows. ■

Proposition 3 *The following identity holds:*

$$\begin{aligned} &\frac{1}{|a|} \sqrt[2]{\frac{z_j}{z_m}} \sqrt[2]{\frac{z_m - z_3}{z_j - z_3}} \frac{1}{\sqrt[2]{z_m - z_3}} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{z_j(1 - z_3)}{z_j - z_3}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}, \frac{z_j}{z_j - z_3} \right) \\ &= \frac{1}{|a|} \sqrt[2]{\frac{z_j}{z_m}} \sqrt[2]{\frac{z_m - z_3}{z_j - z_3}} \frac{1}{\sqrt[2]{z_m - z_3}} \times \left\{ \frac{-z_3}{1 - z_3} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j(1 - z_3)}{z_j - z_3}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}, \frac{z_j}{z_j - z_3} \right) + \right. \\ &\quad \left. \frac{1}{1 - z_3} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j(z_m - z_3)}{z_m(z_j - z_3)}, \frac{z_j}{z_j - z_3} \right) \right\} \end{aligned} \quad (131)$$

Proof. We start with the integral representation of Lauricella's hypergeometric

function:

$$\begin{aligned}
F_D \left(\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, x_1, x_2, x_3 \right) &= \int_0^1 du u^{-1/2} (1 - ux_1)^{-1} (1 - ux_2)^{-1/2} (1 - ux_3)^{1/2} \frac{\Gamma(3/2)}{\Gamma(1/2)} \\
&= \frac{\Gamma(3/2)}{\Gamma(1/2)} \int_0^1 \frac{du}{\sqrt[2]{u}} \frac{1}{1 - ux_1} \frac{1}{\sqrt[2]{1 - ux_2}} \frac{1 - ux_3}{\sqrt[2]{1 - ux_3}} \\
&= \frac{\Gamma(3/2)}{\Gamma(1/2)} \left[\int_0^1 \frac{du}{\sqrt[2]{u}(1 - ux_1)} \frac{1}{\sqrt[2]{1 - ux_2}} \frac{1}{\sqrt[2]{1 - ux_3}} - \right. \\
&\quad \left. \int_0^1 \frac{du}{\sqrt[2]{u}(1 - ux_1)} \frac{u x_3}{\sqrt[2]{1 - ux_2}} \frac{1}{\sqrt[2]{1 - ux_3}} \right] \quad (132)
\end{aligned}$$

with $x_1 := \frac{z_j(1-z_3)}{z_j-z_3}$, $x_2 := \frac{z_j}{z_m} \frac{(z_m-z_3)}{(z_j-z_3)}$, $x_3 := \frac{z_j}{z_j-z_3}$. ■

B Time-delay assuming vanishing Λ .

For the time-delay, in the case of vanishing cosmological constant, we derive the equation:

$$ct = \int^r \frac{r^2(r^2 + a^2)}{\pm \Delta \sqrt{R}} dr + \int^r \frac{2GMr}{\pm c^2 \Delta \sqrt{R}} (a^2 - \Phi a) dr + \int^\theta \frac{a^2 \cos^2 \theta d\theta}{\pm \sqrt{\Theta}} \quad (133)$$

In calculating the last angular term in (133) and using the variable $z = \cos^2 \theta$, one of the integrals we need to calculate is:

$$\frac{1}{2} \frac{a^2}{|a|} \int_0^{z_j} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}} \quad (134)$$

Indeed, its calculation in closed analytic form gave us the result:

$$\begin{aligned}
&\frac{1}{2} \frac{a^2}{|a|} \int_0^{z_j} \frac{dz}{\sqrt{z(z_m - z)(z - z_3)}} \\
&= \frac{1}{2} \frac{a^2}{|a|} \frac{z_j^2(z_m - z_j)}{z_m \sqrt{(z_j - z_m)(z_3 - z_j)z_j}} F_1 \left(1, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_j}{z_m}, \frac{z_j}{z_m} \frac{z_m - z_3}{z_j - z_3} \right) \frac{\Gamma(1)\Gamma(3/2)}{\Gamma(5/2)} \quad (135)
\end{aligned}$$

In total we derive for the angular integrals in (133):

$$\begin{aligned}
\int^{\theta} \frac{a^2 \cos^2 \theta d\theta}{\pm \sqrt{\Theta}} &\equiv A^{\text{time-delay}} = \\
&\frac{1}{2} \frac{a^2}{|a|} \frac{(z_m - z_S) z_m}{\sqrt{z_m(z_m - z_S)(z_m - z_3)}} F_1 \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_S}{z_m}, \frac{z_m - z_S}{z_m - z_3} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} + \\
&[1 - \text{sign}(\theta_S \circ \theta_{mS})] \frac{1}{2} \frac{a^2}{|a|} \frac{z_S^2(z_m - z_S)}{z_m \sqrt{(z_S - z_m)(z_3 - z_S)z_S}} \\
&\times F_1 \left(1, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_S}{z_m}, \frac{z_S}{z_m} \frac{z_m - z_3}{z_S - z_3} \right) \frac{\Gamma(1)\Gamma(3/2)}{\Gamma(5/2)} \\
&+ \frac{1}{2} \frac{a^2}{|a|} \frac{(z_m - z_O) z_m}{\sqrt{z_m(z_m - z_O)(z_m - z_3)}} F_1 \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_m - z_O}{z_m}, \frac{z_m - z_O}{z_m - z_3} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} + \\
&[1 - \text{sign}(\theta_O \circ \theta_{mO})] \frac{1}{2} \frac{a^2}{|a|} \frac{z_O^2(z_m - z_O)}{z_m \sqrt{(z_O - z_m)(z_3 - z_O)z_O}} \\
&\times F_1 \left(1, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{z_O}{z_m}, \frac{z_O}{z_m} \frac{z_m - z_3}{z_O - z_3} \right) \frac{\Gamma(1)\Gamma(3/2)}{\Gamma(5/2)} \\
&+ 2(m-1) \frac{1}{2} \frac{a^2}{|a|} \frac{z_m^2}{\sqrt{z_m^2(z_m - z_3)}} \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)} F \left(\frac{1}{2}, \frac{1}{2}, 2, \frac{z_m}{z_m - z_3} \right) \quad (136)
\end{aligned}$$

We now turn our attention to the calculation of the radial contribution to time-delay in equation (133). Indeed, the first term can be written:

$$\begin{aligned}
&\int_{\alpha}^{r_S} \frac{r^2(r^2 + a^2)}{\Delta \sqrt{R}} dr \\
&= \int_{\alpha}^{r_S} \frac{r^2 dr}{\sqrt{R}} + \int_{\alpha}^{r_S} \frac{2GM r}{c^2 \sqrt{R}} dr - \int_{\alpha}^{r_S} \frac{2a^2 GM r dr}{c^2 \Delta \sqrt{R}} + \frac{4G^2 M^2}{c^4} \int_{\alpha}^{r_S} \frac{\left(1 - \frac{a^2 - 2GM r}{\Delta}\right)}{\sqrt{R}} dr \quad (137)
\end{aligned}$$

In total for this radial term the exact integration yields the result:

$$\begin{aligned}
\int_{\alpha}^{r_S} \frac{r^2(r^2 + a^2)}{\Delta\sqrt{R}} dr &= \frac{\alpha^2 \Omega'' \mathbf{z}_S}{\sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \\
&\times F_D \left(\frac{1}{2}, -2, 2, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta r_S - \alpha}{\alpha r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \\
&+ \frac{\alpha \Omega'' \mathbf{z}_S}{\sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \frac{2GM}{c^2} F_D \left(\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta r_S - \alpha}{\alpha r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{1}{\omega} \frac{r_S - \alpha}{r_S - \beta} \right) 2 \\
&- \frac{\mathbf{z}_S \Omega'' A_+^{td}}{(r_+ - \alpha) \sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \\
&\times \left\{ F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - r_+ r_S - \alpha}{\alpha - r_+ r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \right. \\
&- \left. \frac{r_S - \alpha}{r_S - \beta} F_D \left(\frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{\beta - r_+ r_S - \alpha}{\alpha - r_+ r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\} \\
&- \frac{\mathbf{z}_S \Omega'' A_-^{td}}{(r_- - \alpha) \sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \\
&\times \left\{ F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - r_- r_S - \alpha}{\alpha - r_- r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \right. \\
&- \left. \frac{r_S - \alpha}{r_S - \beta} F_D \left(\frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{\beta - r_- r_S - \alpha}{\alpha - r_- r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\} \\
&+ \frac{4G^2 M^2}{c^4} \frac{\Omega'' \mathbf{z}_S}{\sqrt{\frac{r_S - \alpha}{r_S - \beta}}} F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \\
&- \frac{4G^2 M^2}{c^4} A_{+1}^{td} \frac{\Omega''(-\mathbf{z}_S)}{(r_+ - \alpha) \sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \\
&\times \left\{ F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - r_+ r_S - \alpha}{\alpha - r_+ r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \right. \\
&- \left. \frac{r_S - \alpha}{r_S - \beta} F_D \left(\frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{\beta - r_+ r_S - \alpha}{\alpha - r_+ r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\} \\
&- \frac{4G^2 M^2}{c^4} A_{-1}^{td} \frac{\Omega''(-\mathbf{z}_S)}{(r_- - \alpha) \sqrt{\frac{r_S - \alpha}{r_S - \beta}}} \\
&\times \left\{ F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{\beta - r_- r_S - \alpha}{\alpha - r_- r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \right. \\
&- \left. \frac{r_S - \alpha}{r_S - \beta} F_D \left(\frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{\beta - r_- r_S - \alpha}{\alpha - r_- r_S - \beta}, \frac{r_S - \alpha}{r_S - \beta} \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\delta - \beta r_S - \alpha}{\delta - \alpha r_S - \beta} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} \right\}
\end{aligned}$$

References

- [1] J. Soldner, Ueber die Ablenkung eines Lichtstrahls von seiner geradlinigen Bewegung durch die Attraktion eines Weltkoerpers, an welchem er nahe vorbeigeht, *Berliner Astron. Jahrb.* 1804, 161-172
- [2] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* p831 (1915)
- [3] A. Einstein, *Science* 84, 506 (1936)
- [4] Hans. C. Ohanian, *American J. Physics.* 55(5) 1987, 428-432
- [5] P. Schneider, J. Ehlers, E.E. Falco *Gravitational Lenses*,
- [6] G. V. Kraniotis, Frame dragging and bending of light in Kerr and Kerr-(anti) de Sitter spacetimes, *Class. Quantum Grav.* **22** (2005) 4391-4424
- [7] Ghez, A. M. *et al*, Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits, *ApJ* **689** (2008) 1044, arXiv:0808.2870; Ghez, A. M. *et al*, *ApJ* **586** (2003) L127-31; Ghez, A. M. *et al*, The Galactic Center: A Laboratory for Fundamental Astrophysics and Galactic Nuclei, *An Astro2010 Science White Paper*, arXiv:0903.0383v1 [astro-ph.GA]; Ghez, A. M. *et al*, Increasing the Scientific Return of Stellar Orbits at the Galactic Centre, arXiv:1002.1729 [astro-ph.GA]
- [8] Eisenhauer F *et al*, astro-ph/0508607, 2005, S. Gillessen *et al*, arXiv: 1007.1612 (2010), Genzel R. *et al* arXiv:1006.6064
- [9] C. T. Cunningham and J. M. Bardeen, *Astroph.Journal* 183, (1973) 237
- [10] Bray I., *Phys.Rev.D.* **34** (1986) 367; M. Sereno and F. De Luca, *Phys.Rev.D.* **74** (2006) 123009, arXiv:astro-ph/0609435v2; V. Bozza, F. De Luca and G. Scarpetta, *Phys.Rev.D.* **74** (2006) 063001; V. Bozza, *Phys.Rev.D.* **78** (2008) 063014
- [11] S. E. Vazquez and E. P. Esteban, *Nuov.Com.* 119 B (2004) 489
- [12] Z. Stuchlík and M. Calvani, *Gen.Rel.Grav.* **23** (1991) 507-519; P. Slany and Z. Stuchlík, *Class.Quantum Grav.* **22** (05) 3623-3651
- [13] B. Carter, *Commun.Math.Phys.* **10** (1968) 280-310; M. Demianski, *Acta Astron.* **23** (1973) 197-231; S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, *Phys.Rev.D* **59** (1999) 064005
- [14] E. Teo, *Gen. Rel. Grav.* **35** (2003) 1909
- [15] G. V. Kraniotis, Work in Preparation.

- [16] K. Lake, arXiv:0711.0673v2[gr-qc]
- [17] M. Sereno, *Phys.Rev.D* **77** (2008) 043004
- [18] W. Rindler and M. Ishak, *Phys.Rev.D* **76** (2007) 043006
- [19] G. V. Kraniotis, *Classical and Quantum Gravity* **24** (2007) 1775-1808, arXiv:gr-qc/0602056, C. M. Will, *ApJ*, 674 (2008) L125, D. Merrit, T. Alexander, S. Mikkola and C. M. Will, *Phys. Rev.D* 81(2010) 062002, L. Iorio, arXiv:1008.1720v4[gr-qc], also: Jaroszynski M. *Acta Astronomica* (1998) 48, 653, G. F. Rubilar and A. Echart (2001) *A&A* 374, 95, P.C. Fragile and G. J. Mathews 2000, *ApJ* 542, 328, N. N. Weinberg, M. Milosavljevic and A. M. Ghez, (2005) *ApJ* 622, 878, Preto M. and P. Saha (2009) *ApJ* 703, 1743.